Numerical experiments with plectic Stark–Heegner points

LFANT Seminar

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The Hasse-Weil L-function

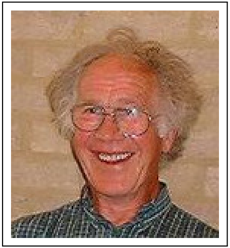
- Let F be a number field.
- Let $E_{/F}$ be an elliptic curve of conductor $\mathfrak{N} = \mathfrak{N}_E$.
- Let K/F be a quadratic extension of F.
  - Assume for simplicity that $\mathfrak{N}$ is square-free, coprime to $\operatorname{disc}(K/F)$.
- For each prime $\mathfrak{p}$ of K, $a_{\mathfrak{p}}(E) = 1 + |\mathfrak{p}| - \#E(\mathbb{F}_{\mathfrak{p}})$.

Hasse-Weil L-function of the base change of E to K ($\Re(s) \gg 0$)

$$L(E/K,s) = \prod_{\mathfrak{p}|\mathfrak{N}} \left(1 - a_{\mathfrak{p}}|\mathfrak{p}|^{-s}\right)^{-1} \times \prod_{\mathfrak{p}\nmid\mathfrak{N}} \left(1 - a_{\mathfrak{p}}|\mathfrak{p}|^{-s} + |\mathfrak{p}|^{1-2s}\right)^{-1}.$$

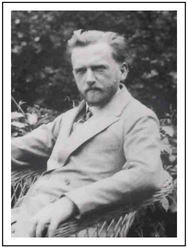
- Modularity conjecture $\Longrightarrow$
  - Analytic continuation of $L(E/K,s)$ to $\mathbb{C}$.
  - Functional equation relating $s \leftrightarrow 2-s$.

The BSD conjecture and Heegner points



Brian Birch

Sir P. Swinnerton-Dyer



Kurt Heegner

Coarse version of BSD conjecture

$$\operatorname{ord}_{s=1} L(E/K, s) = \operatorname{rk}_{\mathbb{Z}} E(K).$$

Heegner Points

- Only for F totally real and K/F totally complex (CM extension).
- Simplest setting: $F = \mathbb{Q}$ (and $K/\mathbb{Q}$ imaginary quadratic), and $\ell \mid \mathfrak{N} \implies \ell$ split in K.

Heegner Points ($K/\mathbb{Q}$ imaginary quadratic)

- $\Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon \mathfrak{N} \mid c\}$.
- Attach to E a modular form:

$$f_E(z) = \sum_{n \geqslant 1} a_n e^{2\pi i n z} \in S_2(\Gamma_0(\mathfrak{N})).$$

- Given $\tau \in K \cap \mathcal{H}$, set $J_\tau = \displaystyle\int_\infty^\tau 2\pi i f_E(z) dz \in \mathbb{C}$.
- Well-defined up to the lattice $\Lambda_E = \left\{\int_\gamma 2\pi i f_E(z) dz \mid \gamma \in \mathrm{H}_1\left(\overline{\Gamma_0(\mathfrak{N})\backslash\mathcal{H}}, \mathbb{Z}\right)\right\}$.
  - There exists an isogeny $\eta \colon \mathbb{C}/\Lambda_E \to E(\mathbb{C})$.
  - Set $P_\tau = \eta(J_\tau) \in E(\mathbb{C})$.
- **Fact:** $P_\tau \in E(H_\tau)$, where H_τ/K is a class field attached to τ.

Theorem (Gross–Zagier)

$$P_K = \mathrm{Tr}_{H_\tau/K}(P_\tau) \text{ nontorsion} \iff L'(E/K, 1) \neq 0.$$

Darmon points – history

- $n = \#\{v \mid \infty_F \colon v \text{ splits in } K\}$.
- $S(E,K) = \Big\{ v \mid \mathfrak{N}\infty_F : v \text{ not split in } K \Big\}$.
- Sign of functional equation for $L(E/K,s)$ should be $(-1)^{\#S(E,K)}$.
- Assume that $s = \#S(E,K)$ is **odd.**
- Fix a finite place $\mathfrak{p} \in S(E,K)$.
  - There is also an archimedean version. . .
- **Darmon** ('99): First construction, with $F = \mathbb{Q}$ and $s = 1$.
- **Trifkovic** ('06): F imaginary quadratic, still $s = 1$.
- **Greenberg** ('08): F totally real, arbitrary ramification, and $s \geqslant 1$.
- **Guitart–M.–Sengun** ('14): F of arbitrary signature, arbitrary ramification, and $s \geqslant 1$.
- **Guitart–M.–Molina** ('18): Adelic generalization, removing all restrictions.

Review of Darmon points

- Define a quaternion algebra $B_{/F}$ and a group $\Gamma \subset \mathrm{SL}_2(F_{\mathfrak{p}})$.
  - The group Γ acts (non-discretely) on $\mathcal{H}_{\mathfrak{p}}$.
- Attach to E a **cohomology** class

$$\Phi_E \in \mathrm{H}^n\left(\Gamma, \mathrm{Meas}^0(\mathbb{P}^1(F_{\mathfrak{p}}, \mathbb{Z}))\right).$$

- Attach to each embedding $\psi\colon K \hookrightarrow B$ a **homology** class

$$\Theta_\psi \in \mathrm{H}_n\left(\Gamma, \mathrm{Div}^0 \mathcal{H}_{\mathfrak{p}}\right).$$

  - Well defined up to the image of $\mathrm{H}_{n+1}(\Gamma, \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_n(\Gamma, \mathrm{Div}^0 \mathcal{H}_{\mathfrak{p}})$.
  - Here δ is a connecting homomorphism arising from

$$0 \longrightarrow \mathrm{Div}^0 \mathcal{H}_{\mathfrak{p}} \longrightarrow \mathrm{Div}\, \mathcal{H}_{\mathfrak{p}} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

- **Cap-product** and **integration** on the coefficients yield an element:

$$J_\psi = \langle \Phi_E, \Theta_\psi \rangle \in K_{\mathfrak{p}}^\times.$$

- J_ψ well-defined up to a multiplicative lattice $L = \langle \Phi_E, \delta(\mathrm{H}_{n+1}(\Gamma, \mathbb{Z}))\rangle$.

Conjectures on Darmon points

$$J_\psi = \langle \Phi_E, \Theta_\psi \rangle \in K_\mathfrak{p}^\times / L.$$

Conjecture 1

There is an isogeny $\eta_{\text{Tate}} \colon K_\mathfrak{p}^\times / L \to E(K_\mathfrak{p})$.

- Proven for totally-real fields (Greenberg, Rotger–Longo–Vigni, Spiess, Gehrmann–Rosso).

The Darmon point attached to E and $\psi \colon K \to B$ is:

$$P_\psi = \eta_{\text{Tate}}(J_\psi) \in E(K_\mathfrak{p}).$$

Conjecture 2

1. The local point P_ψ is **global**, and belongs to $E(K^{\text{ab}})$.
2. P_ψ is nontorsion if and only if $L'(E/K, 1) \neq 0$.

- **Predicts** also the **exact number field** over which P_ψ is defined.
- Includes a **Shimura reciprocity law** like that of Heegner points.

The $\{\mathfrak{p}\}$-arithmetic group Γ

- $B_{/F}$ = quaternion algebra with $\mathrm{Ram}(B) = S(E,K) \setminus \{\mathfrak{p}\}$.
- Induces a factorization $\mathfrak{N} = \mathfrak{p}\mathfrak{D}\mathfrak{m}$.
- Set $R_0^B(\mathfrak{p}\mathfrak{m}) \subset R_0^B(\mathfrak{m}) \subset B$, Eichler orders of levels $\mathfrak{p}\mathfrak{m}$ and $\mathfrak{m}$.
- Define $\Gamma_0^B(\mathfrak{p}\mathfrak{m}) = R_0^B(\mathfrak{p}\mathfrak{m})_1^\times$ and $\Gamma_0^B(\mathfrak{m}) = R_0^B(\mathfrak{m})_1^\times$.
- Set
$$\Gamma = \left(R_0^B(\mathfrak{m})[\mathfrak{p}^{-1}]\right)_1^\times .$$
- Fix an embedding $\iota_\mathfrak{p} \colon R_0^B(\mathfrak{m}) \hookrightarrow M_2(\mathbb{Z}_\mathfrak{p})$.

Lemma

$\iota_\mathfrak{p}$ induces bijections

$$\Gamma/\Gamma_0^B(\mathfrak{m}) \cong \mathcal{V}_0, \quad \Gamma/\Gamma_0^B(\mathfrak{p}\mathfrak{m}) \cong \mathcal{E}_0$$

$\mathcal{V}_0$ (resp. $\mathcal{E}_0$) are the even vertices (resp. edges) of the BT tree.

Integration on $\mathcal{H}_\mathfrak{p}$

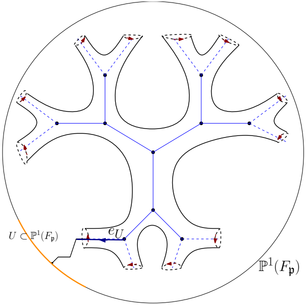
- Let $\mu \in \mathrm{Meas}^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z})$.
- Coleman integration on $\mathcal{H}_\mathfrak{p} = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(F_\mathfrak{p})$ can be defined as:

$$\int_{\tau_1}^{\tau_2} \omega_\mu = \int_{\mathbb{P}^1(F_\mathfrak{p})} \log_\mathfrak{p}\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu(t) = \lim_{\overrightarrow{\mathcal{U}}} \sum_{U \in \mathcal{U}} \log_\mathfrak{p}\left(\frac{t_U - \tau_2}{t_U - \tau_1}\right) \mu(U).$$

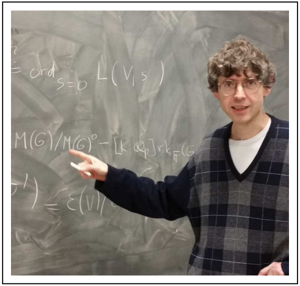
- For $\Gamma \subset \mathrm{PGL}_2(F_\mathfrak{p})$, induce a pairing

$$\mathrm{H}^i(\Gamma, \mathrm{Meas}^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z})) \times \mathrm{H}_i(\Gamma, \mathrm{Div}^0 \mathcal{H}_\mathfrak{p}) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}_\mathfrak{p} .$$

- Bruhat-Tits tree of $\mathrm{GL}_2(F_\mathfrak{p})$, $|\mathfrak{p}| = 2$.
- $\mathcal{H}_\mathfrak{p}$ having the Bruhat-Tits as retract.
- Can identify $\mathrm{Meas}^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z}) \cong \mathrm{HC}(\mathbb{Z})$ $= \{c : \mathcal{E}(\mathcal{T}_\mathfrak{p}) \to \mathbb{Z} \mid \sum_{o(e)=v} c(e) = 0\}$.
- t_U is any point in $U \subset \mathbb{P}^1(F_\mathfrak{p})$.



Plectic conjectures



Jan Nekovář

Tony Scholl

“ $L^{(r)}(E/K,1)$ should be related to CM-points on a r-dimensional quaternionic Shimura variety. ”

Goal : Construct $Q \in \wedge^r(E(K))$ such that

$$Q \text{ non-torsion} \iff L^{(r)}(E/K,1) \neq 0.$$

p-adic Plectic invariants

Michele Fornea

- Let $r \geqslant 1$ with same parity as $\#S(E,K)$.
- $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \subseteq S(E,K)$, $|\mathfrak{p}_i| = p$.
- Let $B_{/F}$ with $\mathrm{Ram}(B) = S(E,K) \smallsetminus S$.
- Set $\Gamma_S = \left(R_0^B(\mathfrak{m})[S^{-1}]\right)_1^\times$.

$F_S = \prod_{\mathfrak{p}\in S} F_\mathfrak{p}$, $\mathbb{P}^1(F_S) = \prod_{\mathfrak{p}\in S} \mathbb{P}^1(F_\mathfrak{p})$, and $\mathcal{H}_S = \prod_{\mathfrak{p}\in S} \mathcal{H}_\mathfrak{p}$.

- Construct $\Phi_E \in \mathrm{H}^n(\Gamma_S, \mathrm{Meas}^0(\mathbb{P}^1(F_S), \mathbb{Z}))$.
  - $\mu\left(\mathbb{P}^1(F_\mathfrak{p}) \times U_{S^\mathfrak{p}}\right) = 0$, for all $\mathfrak{p} \in S$, all $U_{S^\mathfrak{p}} \subseteq \mathbb{P}^1(F_{S^\mathfrak{p}})$.
- Construct $\Theta_\psi \in \mathrm{H}_n(\Gamma_S, \mathbb{Z}_0(\mathcal{H}_S))$.
- Pairing $\mathrm{Meas}^0(\mathbb{P}^1(F_S), \mathbb{Z}) \times \mathrm{Div}^0(\mathcal{H}_S) \to \bigotimes_{\mathfrak{p}\in S} K_\mathfrak{p}$.
  - $\mathrm{H}^n(\Gamma_S, \mathrm{Meas}^0(\mathbb{P}^1(F_S), \mathbb{Z})) \times \mathrm{H}_n(\Gamma_S, \mathbb{Z}_0(\mathcal{H}_S)) \xrightarrow{\langle\cdot,\cdot\rangle} \bigotimes_{\mathfrak{p}\in S} K_\mathfrak{p}$.

Plectic invariant attached to E, K and S

$$J := \langle \Phi_E, \Theta_\psi \rangle \in \bigotimes_{\mathfrak{p}\in S} K_\mathfrak{p}.$$

Cohomology class

- Consider $\varphi_E \in \mathrm{H}^n(\Gamma_0^B(p_S\mathfrak{m}), \mathbb{Z})$ attached to E.
  - Via Eichler–Shimura and Jacquet–Langlands.
- Shapiro isomorphism $\rightsquigarrow \tilde{\varphi}_E \in \mathrm{H}^n(\Gamma_S, \mathrm{coInd}_{\Gamma_0^B(p_S\mathfrak{m})}^{\Gamma_S} \mathbb{Z})$.
- $\mathrm{coInd}_{\Gamma_0^B(p_S\mathfrak{m})}^{\Gamma_S} \mathbb{Z} \cong \mathrm{Maps}(\mathcal{E}(\mathcal{T}_S), \mathbb{Z})$.
- $\mathrm{HC}_S(\mathbb{Z}) = \{c \colon \mathcal{E}(\mathcal{T}_S) \to \mathbb{Z}$ "harmonic in each variable"$\}$:

$$0 \to \mathrm{HC}_S(\mathbb{Z}) \to \mathrm{Maps}(\mathcal{E}(\mathcal{T}_S), \mathbb{Z}) \xrightarrow{\nu} \bigoplus_{\mathfrak{p} \in S} \mathrm{Maps}(\mathcal{V}(\mathcal{T}_\mathfrak{p}) \times \mathcal{E}(\mathcal{T}_{S^\mathfrak{p}}), \mathbb{Z}) \to \cdots$$

- $\mathrm{Meas}^0(\mathbb{P}^1(F_S), \mathbb{Z})$ identified with $\mathrm{HC}_S(\mathbb{Z})$.
- Since φ_E is p-new, have an isomorphism

$$\mathrm{H}^n(\Gamma_S, \mathrm{HC}_S(\mathbb{Z}))_E \cong \mathrm{H}^n(\Gamma_S, \mathrm{Maps}(\mathcal{E}(\mathcal{T}_S), \mathbb{Z}))_E.$$

- Therefore we can define Φ_E, unique up to sign.

Homology class

- Let $\psi\colon \mathcal{O} \hookrightarrow R_0^B(\mathfrak{m})$ be an embedding of an order $\mathcal{O}$ of K.
  - Which is optimal: $\psi(\mathcal{O}) = R_0^B(\mathfrak{m}) \cap \psi(K)$.
- Consider the group $\mathcal{O}_1^\times = \{u \in \mathcal{O}^\times\colon \operatorname{Nm}_{K/F}(u) = 1\}$.
  - $\operatorname{rank}(\mathcal{O}_1^\times) = \operatorname{rank}(\mathcal{O}^\times) - \operatorname{rank}(\mathcal{O}_F^\times) = n$.
- Choose a basis $u_1, \ldots, u_n \in \mathcal{O}_1^\times$ for the non-torsion units.

$$\Delta_\psi = \psi(u_1) \wedge \cdots \wedge \psi(u_n) \in \mathrm{H}_n(\Gamma, \mathbb{Z}).$$

- $K_1^\times$ acts on $\mathcal{H}_S$ through $K_1^\times \overset{\psi}{\hookrightarrow} B_1^\times \overset{\bigoplus_{\mathfrak{p}\in S} \iota_\mathfrak{p}}{\hookrightarrow} \mathrm{SL}_2(F_S)$.
- Let $\tau_\mathfrak{p}, \bar{\tau}_\mathfrak{p}$ be the fixed points of $K_1^\times$ acting on $\mathcal{H}_\mathfrak{p}$.
  - Set $D = \bigotimes_{\mathfrak{p}\in S}(\tau_\mathfrak{p} - \bar{\tau}_\mathfrak{p}) \in \mathbb{Z}_0(\mathcal{H}_S)$.
- Define $\Theta_\psi = [\Delta_\psi \otimes D] \in \mathrm{H}_n(\Gamma_S, \mathbb{Z}_0(\mathcal{H}_S))$.

  Ideally, we'd like to define a class attached to $\bigotimes_{\mathfrak{p}\in S} \tau_\mathfrak{p}$.

Conjectures

- Granting BSD + parity conjectures, expect $r_{\mathsf{alg}}(E/K) \equiv r \pmod 2$.
- Fix embeddings $\iota_{\mathfrak{p}} \colon K \hookrightarrow K_{\mathfrak{p}}$. Get a *regulator* map $\det \colon \wedge^r E(K) \to \hat{E}(K_S), \quad Q_1 \wedge \cdots \wedge Q_r \mapsto \det(\iota_{\mathfrak{p}_i}(Q_j))$.

Conjecture 1 (algebraicity)

Suppose that $r_{\mathsf{alg}}(E/K) \geqslant r$. Then:

- $\exists w \in \wedge^r E(K)$ such that $\eta_{\mathsf{Tate}}(J) = \det(w)$.
- $\eta_{\mathsf{Tate}}(J) \neq 0 \implies r_{\mathsf{alg}}(E/K) = r$.

Conjectures (II)

- Write $T(E) = \{\mathfrak{p} \in S \mid a_{\mathfrak{p}}(E) = 1\}$.
- Set $\rho(E,S) = r_{\text{alg}}(E/F) + |T(E)|$.
- Bergunde–Gehrmann construct a p-adic L-function attached to (E,K,S).
  - Interpolates central L-values of twists of by characters ramified at S.
  - Vanishes to order at least $r(E,K,S) = \max\{\rho(E,S), \rho(E^K,S)\}$.
- Fornea–Gehrmann show that $L_p^{(r(E,K,S))} \doteq J$.
- Assume that $F = \mathbb{Q}(j(E))$.

Conjecture 2 (non-vanishing)

- If $r_{\text{alg}}(E/K) = r = \max\{\rho(E,S), \rho(E^K,S)\}$, then $J \neq 0$.
- If $r_{\text{alg}}(E/K) < r$, then $J \neq 0$ (but don't know arithmetic meaning).
  - Provided that the order of vanishing of L_p allows for it.

Numerical evidence

Joint work with **Xevi Guitart** and **Michele Fornea**.

- We have restricted to F real quadratic of narrow class number one.
  - Therefore take $r = 2$.
- For $\beta \in F$, define $K = F(\sqrt{\beta})$.

Case 1

- We first consider curves E/F where $r_{\mathrm{alg}}(E/F) = 0$.
- Generically, $r_{\mathrm{alg}}(E/K) = 0$ as well.
- Expect J to often be nonzero, unrelated to global points.
- We have checked that this is the case in the following:
  - $F = \mathbb{Q}(\sqrt{13})$, $E =$ 36.1-a2, $\beta = -9w + 8, -12w + 17$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 36.1-a2 , $\beta = -4w + 9$.
- For the following two curves, we have observed $J \simeq 0$ for many β.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$36.1-b1.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$36.1-c1.
- Due to the fact that $a_{\mathfrak{p}_1}(E)a_{\mathfrak{p}_2}(E) = -1 \implies$ extra vanishing of L_p.

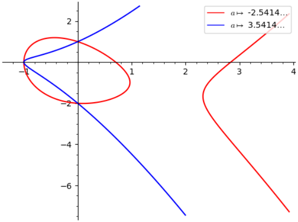
Numerical evidence. Case 2

- We consider curves E/F where $r_{\text{alg}}(E/F) = 1$.
- We impose that $a_{\mathfrak{p}_1}(E)a_{\mathfrak{p}_2}(E) = 1$, so $\max\{\rho(E,S), \rho(E^K,S)\} > 2$.
- Generically, $r_{\text{alg}}(E/K) = 2$.
- In those cases, J should vanish because of an exceptional zero in the p-adic L-function.
- We have checked that this is the case (up to precision p^6) in the following:
  - $F = \mathbb{Q}(\sqrt{13})$, $E =$ 225.1-b2, $\beta = -3w - 1, -12w + 17$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.1-a2, $\beta = -4w + 9$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.1-b1, $\beta = -4w + 9$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.2-a1, $\beta = -3w + 5$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.2-b1, $\beta = -3w + 5$.

Numerical evidence. Case 3

- We consider curves E/F where $r_{\text{alg}}(E/F) = 1$.
- We impose that $a_{\mathfrak{p}_1}(E)a_{\mathfrak{p}_2}(E) = -1$, so $\max\{\rho(E,S), \rho(E^K,S)\} = 2$.
- Generically, $r_{\text{alg}}(E/K) = 2$.
- In those cases, J should be nonzero and related to global points.
- We have checked that this is the case in the following:
  - $F = \mathbb{Q}(\sqrt{13})$, $E =$ 153.2-e2, $\beta = -9w + 8$.
  - $F = \mathbb{Q}(\sqrt{13})$, $E =$ 207.1-c1, $\beta = -9w - 4, -9w + 8$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.1-d1, $\beta = -4w + 9$.
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 63.2-d1, $\beta = -3w + 5$
  - $F = \mathbb{Q}(\sqrt{37})$, $E =$ 99.2-c1, $\beta = -8w + 17, -16w + 9, -20w + 29, -9w + 14, -12w + 29, -32w + 41, -12w - 7, -35w + 17$.
- In one of the examples, we obtain what seems to be zero. We expect that this is due to the low working precision...

A pretty example



$F = \mathbb{Q}(\sqrt{13})$, $w = \frac{1+\sqrt{13}}{2}$,
$E/F : y^2+xy+y = x^3+wx^2+(w+1)\,x+2$,
$K = F(\sqrt{\beta})$, with $\beta = 62 - 21w$.

- $E(K)\otimes\mathbb{Q} = \langle P, Q\rangle$, with
  $P = (3-w, 4-w)$ and $Q = (8 - \frac{25}{9}w, (\frac{-23}{27}w + \frac{17}{6})\sqrt{\beta} + \frac{25}{18}w - \frac{9}{2})$.
- We may compute
  $\log_{E_1}(P_1 - \bar{P}_1)\otimes\log_{E_2}(Q_2 - \bar{Q}_2) - \log_{E_1}(Q_1 - \bar{Q}_1)\otimes\log_{E_2}(P_2 - \bar{P}_2) \in \mathbb{Q}_{p^2}\otimes\mathbb{Q}_{p^2}$.
- Projecting $\mathbb{Q}_{p^2}\otimes\mathbb{Q}_{p^2} \to \mathbb{Q}_p$, get $2\cdot 3^2 + 3^6 + 2\cdot 3^7 + 3^9 + O(3^{10})$.
- This matches our computation of $J = 2\cdot 3^2 + 3^6 + O(3^7)$.

[1] https://www.lmfdb.org/EllipticCurve/2.2.37.1/63.2/d/1

Computation of the cohomology class

- Assume, for concreteness, that $r = 2$.
- We start with $\varphi_E \in \mathrm{H}^1(\Gamma_0(\mathfrak{p}_1\mathfrak{p}_2), \mathbb{Z})$.
- Shapiro isomorphism yields an isomorphism $\mathrm{H}^1(\Gamma_0(\mathfrak{p}_1\mathfrak{p}_2), \mathbb{Z}) \cong \mathrm{H}^1(\Gamma_S, \mathrm{coInd}\,\mathbb{Z})$.
  - $\rightsquigarrow [\tilde{\varphi}_E] \in \mathrm{H}^1(\Gamma_S, \mathrm{coInd}\,\mathbb{Z})$.
  - The exact cocycle representative depends on a choice of coset representatives for $\Gamma_S/\Gamma_0(\mathfrak{p}_1\mathfrak{p}_2)$.
- Have a long-exact sequence

$$\mathrm{H}^1(\Gamma_S, \mathrm{HC}(\mathbb{Z})) \to \mathrm{H}^1(\Gamma_S, \mathrm{coInd}\,\mathbb{Z}) \xrightarrow{\nu} \bigoplus_{\mathfrak{p}\in S} \mathrm{H}^1(\Gamma_S, \mathrm{Maps}(\mathcal{V}(\mathcal{T}_{\mathfrak{p}})\times\mathcal{E}(\mathcal{T}_{S^{\mathfrak{p}}}), \mathbb{Z}))$$

- φ_E is p-new $\rightsquigarrow [\Phi_E] \in \mathrm{H}^1(\Gamma_S, \mathrm{HC}(\mathbb{Z}))$ lifting $[\tilde{\varphi}_E]$.
- When $r = 1$, one can choose appropriate coset representatives (called *radial*), which ensure that $\Phi_E = \tilde{\varphi}_E$.
- We don't know whether there are coset representatives that allow for that in our setting.

Lifting to $\mathrm{H}^1(\Gamma_S, \mathrm{HC}(\mathbb{Z}))$

- We know that $\exists \phi : \mathcal{E}(\mathcal{T}_S) \to \mathbb{Z}$ such that $\tilde{\varphi}_E - \partial\phi \in Z^1(\Gamma_S, \mathrm{HC}(\mathbb{Z}))$.
- First, compute $\nu(\tilde{\varphi}_E) = \partial(f_1, f_2)$,

$$f_1 \colon \mathcal{V}(\mathcal{T}_{\mathfrak{p}_1}) \times \mathcal{E}(\mathcal{T}_{\mathfrak{p}_2}) \to \mathbb{Z}, \quad f_2 \colon \mathcal{E}(\mathcal{T}_{\mathfrak{p}_1}) \times \mathcal{V}(\mathcal{T}_{\mathfrak{p}_2}) \to \mathbb{Z}.$$

- For each $(v, e) \in \mathcal{V}(\mathcal{T}_{\mathfrak{p}_1}) \times \mathcal{E}(\mathcal{T}_{\mathfrak{p}_2})$, pick $\gamma \in \Gamma_S$ such that $\gamma(v, e) = (v_0, e_*)$, with $v_0 \in \{v_*, \hat{v}_*\}$.

$$f_1(v, e) - f_1(v_0, e_*) = \nu_1(\tilde{\varphi}_E(\gamma))(v_0, e_*).$$

- Analogously, $f_2(e, v) - f_2(e_*, v_0) = \nu_2(\tilde{\varphi}_E(\gamma))(e_*, v_0)$.
- Hence the four values $f_1(v_*, e_*)$, $f_1(\hat{v}_*, e_*)$, $f_2(v_*, e_*)$, $f_2(\hat{v}_*, e_*)$ determine all the remaining ones.
- Knowing the functions f_1 and f_2 to some fixed radius allows to find ϕ such that $\nu(\phi) = (f_1, f_2)$, by solving a linear system of equations.

Linear algebra

- To compute ϕ we need to solve a system of:
  - $2\frac{(p+1)(p^d-1)}{p-1}\frac{p^d+p^{d-1}-2}{p-2} = O(p^{2d-1})$ equations, in
  - $\frac{(p+1)^2(p^d-1)^2}{(p-1)^2} = O(p^{2d})$ unknowns.
- $p = 3$, $d = 7$: get $12,740,008$ equations in $19,114,384$ unknowns.
- Luckily, it's sparse: only $p + 1$ unknowns involved in each equation.
- We implemented a custom row reduction, avoiding division and choosing pivots that maintain sparsity.
- Takes ~ 60 hours using 16 CPUs to compute f_1 and f_2.
- Solve the system in ~ 2 hours (non-parallel), using ~ 300GB RAM.
- Integration takes ~ 10 hours using 64 CPUs.

Further work

- So far we can compute invariants attached to differences $\tau_{\mathfrak{p}} - \bar{\tau}_{\mathfrak{p}}$.
  - Fornea–Gehrmann: refined invariants attached to $\tau_{\mathfrak{p}}$, more akin to Darmon points. Effective computation?
- The Riemann sums algorithm runs in exponential time in the precision.
  - Need an overconvergent method to compute the invariants in polynomial time.
- More experiments are needed in other settings (imaginary quadratic, mixed signature).
- To compute plectic Heegner points, need fundamental domains for Bruhat–Tits trees acted on by groups attached to totally definite quaternion algebras (work in progress).

Merci !

http://www.mat.uab.cat/~masdeu/