Apéry-Like Recursions and Modular Forms

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Apéry Recursions

In his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry introduced the following recursions:

$$(n+1)^2u_{n+1} - (11n^2+11n+3)u_n - n^2u_{n-1} = 0$$
$$(n+1)^3u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3u_{n-1} = 0\ ,$$

both with $u_{-1} = 0$, $u_0 = 1$.

Remarkable fact: all the u_n are integers (a priori they could have a denominator $n!^2$ or $n!^3$ respectively), and this plays an essential part in Apéry's proofs.

Second Remarkable fact: when suitably interpreted, in both cases the generating function $\sum_{n\ge0} u_nt^n$ is a modular function (of weight 1 and 2 respectively), fact discovered by F. Beukers.

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Initial Search for Recursions I

Focus first on recursions of degree two, and to simplify shape of differential equation, recursions of the type

$$(n+1)^2 u_{n+1} - (an^2 + an + b)u_n + cn^2 u_{n-1} = 0$$

with $u_{-1} = 0$, $u_0 = 1$, so that $u_1 = b$.

Note changing u_n into u_n/D^n is equivalent to changing (a, b, c) into (Da, Db, D^2c). Thus, may assume that sequence u_n is primitive (no $D > 1$ with $D^n \mid u_n$) and $u_1 \geq 0$ ($D = -1$).

We can do a reasonable search for $(u_1, u_2, u_3) \in \mathbb{Z}^3$ with $u_1 \geq 0$. We note experimentally that this leads to a, b, c all integral (not clear a priori). Thus, loop instead on $(a, b = u_1, u_2) \in \mathbb{Z}^3$.

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Initial Search for Recursions II

After a few minutes search, find a reasonably large number of (possible) primitive solutions, for instance for $|a| \leq 250$, $0 \leq u_1 = b \leq 100$, and $|u_2| \leq 1000$ we find 34 solutions.
Analysis of solutions:

- Terminating sequences: i.e., $u_n = 0$ for n large. Easy to see corresponds to $(a, b, c) = (-1, k(k+1), 0)$ for $k \in \mathbb{Z}_{\geq 1}$. Six sequences in our list. $u_n = \binom{k}{n}\binom{k+n}{n}$, generating function $F(t) = \sum_{n \geq 0} u_n t^n = P_k(1-2t)$, P_k Legendre polynomial.

Example:
$(a, b, c) = (-1, 20, 0)$: $u = (1, 20, 90, 140, 70, 0, 0, 0, \dots)$

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Initial Search for Recursions III

• More general Hypergeometric solutions: $c = 0$, so u_{n+1}/u_n is a simple rational function. Not all give integral solutions: need $(a, b, c) = (-Qq^2, Qp(p+q), 0)$ with $\gcd(p, q) = 1$, $q > 0$, and $Q = \prod_{\ell | q} \ell^{\lceil 2/(\ell-1) \rceil}$ (note: dividing by Qq^2 gives again $(-1, k(k+1), 0)$ with $k = p/q$). Eleven additional sequences among our list.

Example:
$(a, b, c) = (16, 4, 0)$: $u = (1, 4, 36, 400, 4900, 63504, \dots)$

Initial Search for Recursions IV

• Polynomial solutions, i.e., u_n is a polynomial in n. Easy to show by identification of leading coefficients in recursion that $(a, b, c) = (2, k^2 + k + 1, 1)$. Eight more sequences.

Example:
$(a, b, c) = (2, 7, 1)$: $u = (1, 7, 19, 37, 61, 91, 127, \dots)$

Initial Search for Recursions V

• Once again replacing k by p/q and scaling leads to $(a,b,c)=(2Qq^2,Q(p^2+pq+q^2),Q^2q^4)$, which Zagier calls Legendrian sequences. Three more sequences.

Example:
$(a,b,c)=(32,12,256)$: $u=(1,12,164,2352,34596,\dots)$

We have thus explained 28 out of the 34 sequences found, and all the above families are infinite and trivially parametrized. There remains six unexplained sequences which we thus call sporadic. A much larger search for several hours does not give any additional sequences than the four infinite families plus the six sporadic sequences.

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We have thus explained 28 out of the 34 sequences found, and all the above families are infinite and trivially parametrized. There remains six unexplained sequences which we thus call sporadic. A much larger search for several hours does not give any additional sequences than the four infinite families plus the six sporadic sequences.

The six sporadic solutions are:

$(a, b, c) = (7, 2, -8)$: $u = (1, 2, 10, 56, 346, 2252, \dots)$.

$(a, b, c) = (9, 3, 27)$: $u = (1, 3, 8, 21, 9, -297, \dots)$.

$(a, b, c) = (10, 3, 9)$: $u = (1, 3, 15, 93, 639, 4653, \dots)$.

$(a, b, c) = (11, 3, -1)$: $u = (1, 3, 19, 147, 1251, 11253, \dots)$ (Apéry's sequence).

$(a, b, c) = (12, 4, 32)$: $u = (1, 4, 20, 112, 676, 4304, \dots)$.

$(a, b, c) = (17, 6, 72)$: $u = (1, 6, 42, 312, 2394, 18756, \dots)$.

Auxiliary Sequences I

In each case can define an auxiliary sequence v_n with $v_0 = 0$ and $v_1 = 1$ and the same recursion, and look at the convergence of v_n/u_n. For the four infinite families, either nonconvergent or slow convergent with known limits. Since same recursion, explicit continued fraction.

Other surprising fact: like in Apéry, all these auxiliary v_n have a denominator which does not grow like $n!^2$, but only like d_n^2 (essentially e^{2n}), where $d_n = \mathrm{lcm}(1, 2, ..., n)$.

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Auxiliary Sequences II

For the continued fraction corresponding to the six sporadic sequences, five converge, and exponentially fast (like $1/\alpha^n$ with $\alpha = (a + \sqrt{a^2 - 4c})^2/(4c)$) to a rational number times $\zeta(2)$, NOT, $L(\chi_{-3}, 2)$, $\zeta(2)$, $L(\chi_{-4}, 2)$, and $L(\chi_{-3}, 2)$ respectively, but unfortunately only the Apéry sequence for $\zeta(2)$ proves irrationality (needs convergence at least in e^{4n}); note that irrationality of $L(\chi_D, 2)$ with $D < 0$ is unknown.

Auxiliary Sequences III

However, all five give nice continued fractions. In addition to Apéry's continued fraction for $\zeta(2)$ we have

$$L(\chi_{-3},2)=\cfrac{2}{P(1)-\cfrac{9\cdot 1^4}{P(2)-\cfrac{9\cdot 2^4}{P(3)-\ddots}}}$$

with $P(n)=10n^2-10n+3$ (convergence in 9^{-n}), and

$$L(\chi_{-4},2)=\cfrac{1/2}{P(1)-\cfrac{2\cdot 1^4}{P(2)-\cfrac{2\cdot 2^4}{P(3)-\ddots}}}$$

with $P(n)=3n^2-3n+1$ (convergence in 2^{-n}).

Modular Properties I

Important theorem: if $t(\tau)$ is (nonconstant) modular of weight 0 and $f(\tau)$ modular of weight k, then locally (for instance around $\tau = i\infty$) if one expresses f in terms of t as $f(\tau) = F(t(\tau))$, then F satisfies a linear differential equation of order $k+1$ with algebraic coefficients, and even polynomial coefficients if t is a Hauptmodul, i.e., generates the field of modular functions.

We prove this in weight $k = 1$ because we need the DE.

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Modular Properties II

Thus, let $t(\tau)$ be modular of weight 0 and $f(\tau)$ modular of weight 1. Let as usual $D = (1/(2\pi i))d/d\tau = qd/dq$ with $q = e^{2\pi i\tau}$. Then $D(t)/f^2$ is modular of weight 0, and since the field of modular functions has transcendence degree 1, there exists an algebraic function α such that $D(t)/f^2 = \alpha(t)$.

Similarly, one checks that $2D(f)^2 - fD^2(f)$ is modular of weight 6 (essentially equal to the RC bracket $[f,f]_2$), so there exists an algebraic function β with $(2D(f)^2 - fD^2(f))/(f^4D(t)) = \beta(t)$.

Immediate computation then shows $\alpha dF/dt = D(t)/f^2$, then $(d/dt)(\alpha dF/dt) = -F(t)\beta(t)$, so DE, where $F' = dF/dt$:

$$(\alpha F')' + \beta F = 0 \; .$$

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Exactly of the above form with $\alpha(t)=t(1-at+ct^2)$ and $\beta(t)=-b+ct$.

Note $D(t)/(\alpha(t)F(t(\tau))^2)=1$ and $D(t)=(dt/d\tau)/(2\pi i)$, so $2\pi i\tau=\int dt/(\alpha(t)F(t)^2)$. In our case $\alpha(t)=t+O(t^2)$ and $F(t)=1+O(t)$, so

$$2\pi i\tau=\int_0^t\left(\frac{1}{\alpha(x)F(x)^2}-\frac{1}{x}\right)dx+\log(Ct)$$

for some constant C.

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Modular Properties IV

We may choose t so that $t(\tau) = q + O(q^2)$ so $C = 1$ and

$$q = t \exp\left(\int_0^t \left(\frac{F(x)^{-2}}{1 - ax + cx^2} - 1\right) \frac{dx}{x}\right) .$$

Using $F(x) = 1 + bx + O(x^2)$ we find
$q = t + (a - 2b)t^2 + O(t^3)$, this can be inverted $t = T(q)$, hence
$f = F(T(q))$ is our desired modular function of weight 1.

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Modular Properties V

Possible Pari/GP script:

```
findmodular(a,b,c,L=16)=
{ my(V=vector(L+1),un=1,unm1=0,unp1,F,t,f);

  V[1]=1;
  for(n=0,L-1,
    unp1=((a*n*(n+1)+b)*un-c*n^2*unm1)/(n+1)^2;
    unm1=un;un=unp1;V[n+2]=un
  );
  F=Ser(V);
  t=serreverse(x*exp(intformal((1/(F^2*(1-a*x+c*x^2))-1)/x
  f=subst(F,x,t);
  [t,f];
}
```

Modular Example I

First sporadic example: $(a, b, c) = (7, 2, -8)$, we find

$$t = x - 3x^2 + 3x^3 + 5x^4 - 18x^5 + 15x^6 + 24x^7 - 75x^8 + 57x^9 + \cdots$$
$$f = 1 + 2x + 4x^2 + 2x^3 + 2x^4 + 4x^6 + 4x^7 + 4x^8 + 2x^9 + \cdots$$

Easily recognized as eta quotients

$$t(\tau) = \frac{\eta(\tau)^3\eta(6\tau)^9}{\eta(2\tau)^3\eta(3\tau)^9} \quad \text{and} \quad f(\tau) = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2\eta(6\tau)^3}\ .$$

In this way, we find that 12 out of our initial 34 sequences (including all six sporadic ones) have a similar modular interpretation, but not necessarily as eta quotients.

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Modular Example II

For instance, for Apéry's example we find that

$$t(\tau) = q \prod_{n \geq 1} (1 - q^n)^{5\left(\frac{n}{5}\right)}$$

(which is not an eta quotient but satisfies the degree two algebraic equation $(1 - 11t - t^2)/t = (\eta(\tau)/\eta(5\tau))^6$), and

$$f^2(\tau) = \frac{\eta(5\tau)^5}{\eta(\tau)t(\tau)}\ .$$

Degree Three Recursions I

Previous search generalized Apéry recursion for $\zeta(2)$. We now generalize Apéry recursion for $\zeta(3)$. Consider degree three recursions of following specific shape (can be slightly more general, see below):

$$(n+1)^3 u_{n+1} - (2n+1)(an^2 + an + b)u_n + cn^3 u_{n-1} = 0 ,$$

again with $u_{-1} = 0$, $u_0 = 1$, so $u_1 = b$. As before, small search on $(u_1, u_2, u_3) \in \mathbb{Z}^3$ implies $(a, b, c) \in \mathbb{Z}^3$ (with one trivial exception $(a, b, c) = (-1/3, 2, 0)$ which gives the terminating sequence $u = (1, 2, 1, 0, 0, 0, \dots)$), so again we loop on $(a, b = u_1, u_2) \in \mathbb{Z}^3$ with $b \geq 0$.

After looping for $|a| \leq 500$, $0 \leq b \leq 120$, and $|c| \leq 4000$ we find 31 solutions, and easily check that we have 4 Terminating, 9 Hypergeometric, 7 Polynomial, and 5 Legendrian sequences, leaving 6 sporadic solutions, and no more after a much larger search.

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Degree Three Recursions II

The six sporadic solutions are:

$(a,b,c)=(7,3,81)$: $u=(1,3,9,3,-279,-2997,\dots)$.

$(a,b,c)=(9,3,-27)$: $u=(1,3,27,309,4059,57753,\dots)$.

$(a,b,c)=(10,4,64)$: $u=(1,4,28,256,2716,31504,\dots)$.

$(a,b,c)=(11,5,125)$: $u=(1,5,35,275,2275,19255,\dots)$.

$(a,b,c)=(12,4,16)$: $u=(1,4,40,544,8536,145504,\dots)$.

$(a,b,c)=(17,5,1)$: $u=(1,5,73,1445,33001,819005,\dots)$.
(Apéry's sequence).

Degree Three Recursions III

Once again we can define an auxiliary sequence v_n with $v_0 = 0$ and $v_1 = 1$ and the same recursion, and look at the convergence of v_n/u_n. For the four infinite families, either nonconvergent or slow convergent with known limits. Again the denominator of v_n does not grow too fast, like $d_n^3 \approx e^{3n}$.

For the continued fractions associated with the six sporadic solutions, four converge, and exponentially fast, to a rational number times NOT, $\pi^3\sqrt{3}$, $\zeta(3)$, $\zeta(3)$, NOT, and $\zeta(3)$ respectively, but again unfortunately only the Apéry sequence for $\zeta(3)$ proves irrationality (that of $\pi^3\sqrt{3}$ is of course well-known).

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Degree Three Recursions IV

Note nice continued fraction for $\pi^3\sqrt{3}$:

$$\frac{4\pi^3\sqrt{3}}{243} = \cfrac{1}{P(1) + \cfrac{3\cdot 1^6}{P(2) + \cfrac{3\cdot 2^6}{P(3) + \ddots}}}$$

with $P(n) = 6n^3 - 9n^2 + 5n - 1$.

Similar to the Apéry continued fraction for $\zeta(3)$:

$$\frac{\zeta(3)}{6} = \cfrac{1}{P(1) - \cfrac{1^6}{P(2) - \cfrac{2^6}{P(3) - \ddots}}}$$

with $P(n) = 34n^3 - 51n^2 + 27n - 5$.

Degree Three Recursions IV

Note nice continued fraction for $\pi^3\sqrt{3}$:

$$\frac{4\pi^3\sqrt{3}}{243}=\cfrac{1}{P(1)+\cfrac{3\cdot 1^6}{P(2)+\cfrac{3\cdot 2^6}{P(3)+\ddots}}}$$

with $P(n)=6n^3-9n^2+5n-1$.

Similar to the Apéry continued fraction for $\zeta(3)$:

$$\frac{\zeta(3)}{6}=\cfrac{1}{P(1)-\cfrac{1^6}{P(2)-\cfrac{2^6}{P(3)-\ddots}}}$$

with $P(n)=34n^3-51n^2+27n-5$.

Recursions of degree two correspond to modular forms of weight 1, and those of degree three to modular forms of weight 2. More difficult. However, amazing identity discovered rather recently. For instance, look again at the six sporadic (a, b, c) in degree two:

$(a_2, b_2, c_2) = (7, 2, -8)$, $(9, 3, 27)$, $(10, 3, 9)$, $(11, 3, -1)$, $(12, 4, 32)$, and $(17, 6, 72)$.

and in degree three:
$(a_3, b_3, c_3) = (7, 3, 81)$, $(9, 3, -27)$, $(10, 4, 64)$, $(11, 5, 125)$, $(12, 4, 16)$, and $(17, 5, 1)$.

Notice immediately that $a_3 = a_2$, almost immediately that $b_3 = a_2 - 2b_2$, and that $c_3 = a_2^2 - 4c_2$.

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Modular Properties II

Remarkable identity proved by G. Almkvist, D. van Straten, and W. Zudilin:
Assume u_n degree two as above, i.e., $u_{-1} = 0$, $u_0 = 1$, and $(n+1)^2 u_{n+1} - (an^2 + an + b)u_n + cn^2 u_{n-1} = 0$, and $U(t) = \sum_{n \geq 0} u_n t^n$ generating function.

Define a sequence w_n of degree three by $w_{-1} = 0$, $w_0 = 1$, and

$$(n+1)^3 w_{n+1} - (2n+1)(an^2 + an + a - 2b)w_n + (a^2 - 4c)n^3 w_{n-1} = 0 \ ,$$

and $W(t) = \sum_{n \geq 0} w_n t^n$ generating function. Identity

$$U(t)^2 = \frac{1}{1 - at + ct^2} W\left(\frac{-t}{1 - at + ct^2}\right) \ .$$

Modular Properties III

Note: this is similar to Clausen identity of the shape ${}_2F_1^2 = {}_3F_2$ since weight 1 corresponds to ${}_2F_1$ and weight 2 to ${}_3F_2$.

Proved exactly in the same way: show that both sides satisfy the same linear differential equation of order three with same initial conditions. Clausen can be proved in a few lines. The above needs 2 pages for the complete details, or the use of a CAS.

Modular Properties IV

Integrality properties are essentially equivalent. Surprising consequence: up to scaling, all recursions of degree three follow from those of degree two and the above theorem. In particular, Apéry's recursion for $\zeta(3)$ follows from a much simpler one in degree two.

Other consequence: can immediately deduce modular parametrizations of degree three recursions from those of degree two. For instance, if u_n is the Apéry sequence for $\zeta(3)$, we have

$$\sum_{n\geq 0} u_n \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12n} = \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5}\,.$$

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More General Degree Three Recursions I

S. Cooper has suggested the study of the slightly more general recursion

$$(n+1)^3 w_{n+1} - (2n+1)(an^2+an+b)w_n + (cn^3+dn)w_{n-1} = 0\ ,$$

again with $u_{-1} = 0$, $u_0 = 1$, so $u_1 = b$. Motivation: if u_n satisfies the usual degree two recursion as before, then $w_n = \binom{2n}{n} u_n$ satisfies

$$(n+1)^3 w_{n+1} - (2n+1)(2an^2+2an+2b)w_n + (16cn^3-4cn)w_{n-1} = 0\ ,$$

and there is a similar Clausen-type identity for $\sum_{n\geq 0} w_n t^n$.

More General Degree Three Recursions II

A similar search finds two additional sporadic sequences and more modular parametrizations. This gives for instance the following CF:

$$\pi^2 = \cfrac{42}{P(1) + \cfrac{1^3 \cdot 2 \cdot 3 \cdot 4}{P(2) + \cfrac{2^3 \cdot 5 \cdot 6 \cdot 7}{P(3) + \cfrac{3^3 \cdot 8 \cdot 9 \cdot 10}{P(4) + \ddots}}}}$$

with $P(n) = 26n^3 - 39n^2 + 21n - 4$.

Note that this combined with the arithmetic properties of u_n and v_n proves irrationality of π^2 with a better irrationality measure than Apéry's initial continued fraction.

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Thank you for your attention.