Numerics of classical elliptic functions, elliptic integrals and modular forms

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Introduction

Elliptic functions

- $F(z+\omega_1 m+\omega_2 n) = F(z), \quad m,n\in\mathbb{Z}$
- Can assume $\omega_1 = 1$ and $\omega_2 = \tau \in \mathbb{H} = \{\tau\in\mathbb{C} : \mathrm{Im}(\tau) > 0\}$

Elliptic integrals

- $\int R(x, \sqrt{P(x)})dx$; inverses of elliptic functions

Modular forms/functions on $\mathbb{H}$

- $F(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k F(\tau)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$
- Related to elliptic functions with fixed z and varying lattice parameter $\omega_2/\omega_1 = \tau \in \mathbb{H}$

Jacobi theta functions (quasi-elliptic functions)

- Used to construct elliptic and modular functions

Numerical evaluation

Lots of existing literature, software (Pari/GP, Sage, Maple, Mathematica, Matlab, Maxima, GSL, NAG, . . .).

This talk will mostly review standard techniques (and many techniques will be omitted).

My goal: general purpose methods with

- Rigorous error bounds
- Arbitrary precision
- Complex variables

Implementations in the C library Arb (`http://arblib.org/`)

Why arbitrary precision?

Applications:

- Mitigating roundoff error for lengthy calculations
- Surviving cancellation between exponentially large terms
- High order numerical differentiation, extrapolation
- Computing discrete data (integer coefficients)
- Integer relation searches (LLL/PSLQ)
- Heuristic equality testing

Also:

- Can increase precision if error bounds are too pessimistic

Most interesting range: $10 - 10^5$ digits. (Millions, billions...?)

Ball/interval arithmetic

A real number in Arb is represented by a rigorous enclosure as a high-precision midpoint and a low-precision radius:

$$[3.14159265358979323846264338328 \pm 1.07 \cdot 10^{-30}]$$

Complex numbers: $[m_1 \pm r_1] + [m_2 \pm r_2]i$.

Key points:

- Error bounds are propagated automatically
- As cheap as arbitrary-precision floating-point
- To compute $f(x) = \sum_{k=0}^{\infty} \square \approx \sum_{k=0}^{N-1} \square$ rigorously, only need analysis to bound $|\sum_{k=N}^{\infty} \square|$
- Dependencies between variables may lead to inflated enclosures. Useful technique is to compute $f([m \pm r])$ as $[f(m) \pm s]$ where $s = |r| \sup_{|x-m| \le r} |f'(x)|$.

Reliable numerical evaluation

Example: $\sin(\pi + 10^{-35})$

IEEE 754 double precision result: `1.2246467991473532e-16`

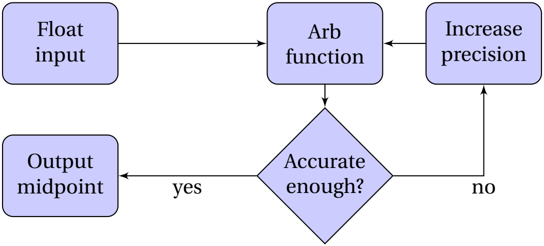
Adaptive numerical evaluation with Arb:

64 bits: $[\pm 6.01 \cdot 10^{-19}]$

128 bits: $[-1.0 \cdot 10^{-35} \pm 3.38 \cdot 10^{-38}]$

192 bits: $[-1.00000000000000000000 \cdot 10^{-35} \pm 1.59 \cdot 10^{-57}]$

Can be used to implement reliable floating-point functions, even if you don't use interval arithmetic externally:



Elliptic and modular functions in Arb

- $PSL_2(\mathbb{Z})$ transformations and argument reduction
- Jacobi theta functions $\theta_1(z,\tau),\ldots,\theta_4(z,\tau)$
- Arbitrary z-derivatives of Jacobi theta functions
- Weierstrass elliptic functions $\wp^{(n)}(z,\tau), \wp^{-1}(z,\tau), \zeta(z,\tau), \sigma(z,\tau)$
- Modular forms and functions: $j(\tau), \eta(\tau), \Delta(\tau), \lambda(\tau), G_{2k}(\tau)$
- Legendre complete elliptic integrals $K(m), E(m), \Pi(n,m)$
- Incomplete elliptic integrals $F(\phi,m), E(\phi,m), \Pi(n,\phi,m)$
- Carlson incomplete elliptic integrals R_F, R_J, R_C, R_D, R_G

Possible future projects:

- The suite of Jacobi elliptic functions and integrals
- Asymptotic complexity improvements

An application: Hilbert class polynomials

For $D < 0$ congruent to 0 or 1 mod 4,

$$H_D(x) = \prod_{(a,b,c)} \left(x - j\left(\frac{-b+\sqrt{D}}{2a} \right) \right) \in \mathbb{Z}[x]$$

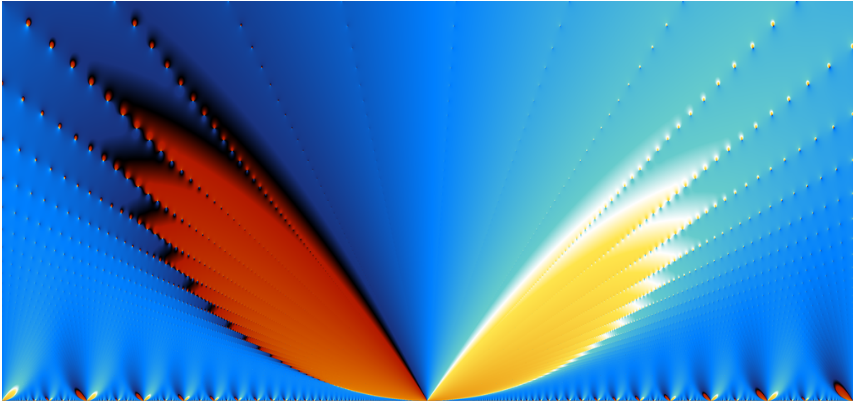
where (a, b, c) is taken over all the primitive reduced binary quadratic forms $ax^2 + bxy + cy^2$ with $b^2 - 4ac = D$.

Example:
$H_{-31} = x^3 + 39491307x^2 - 58682638134x + 1566028350940383$

Algorithms: modular, complex analytic

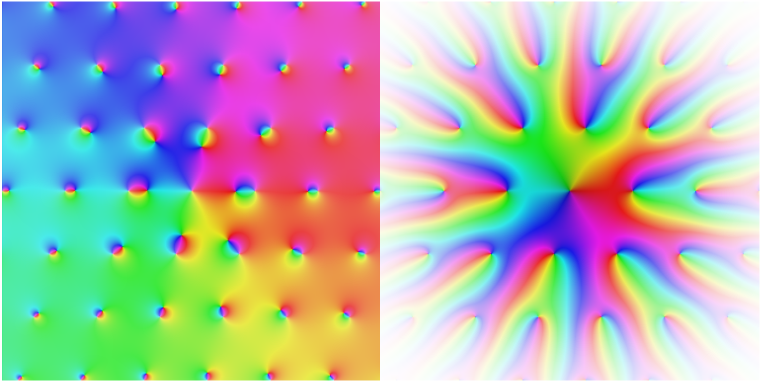
$-D$	Degree	Bits	Pari/GP	classpoly	CM	Arb
$10^6 + 3$	105	8527	12 s	0.8 s	0.4 s	0.14 s
$10^7 + 3$	706	50889	194 s	8 s	29 s	17 s
$10^8 + 3$	1702	153095	1855 s	82 s	436 s	274 s

Some visualizations



The Weierstrass zeta-function $\zeta(0.25 + 2.25i, \tau)$ as the lattice parameter τ varies over $[-0.25, 0.25] + [0, 0.15]i$.

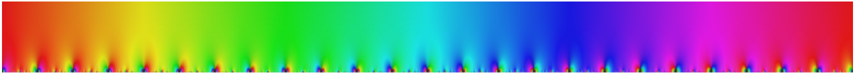
Some visualizations



The Weierstrass elliptic functions $\zeta(z, 0.25 + i)$ (left) and $\sigma(z, 0.25 + i)$ (right) as z varies over $[-\pi, \pi], [-\pi, \pi]i$.

Some visualizations

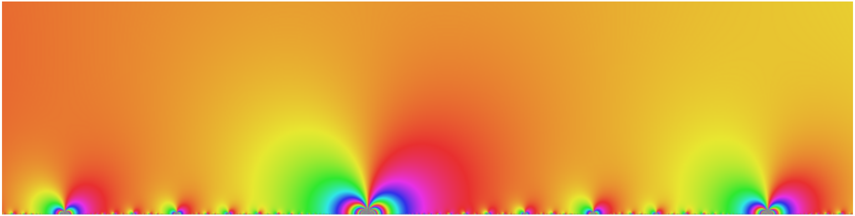
The function $j(\tau)$ on the complex interval $[-2, 2] + [0, 1]i$.



The function $\eta(\tau)$ on the complex interval $[0, 24] + [0, 1]i$.

Some visualizations

Plot of $j(\tau)$ on $[\sqrt{13}, \sqrt{13} + 10^{-101}] + [0, 2.5 \times 10^{-102}]i$.



Plot of $\eta(\tau)$ on $[\sqrt{2}, \sqrt{2} + 10^{-101}] + [0, 2.5 \times 10^{-102}]i$.

Approaches to computing special functions

- Numerical integration (integral representations, ODEs)
- **Functional equations (argument reduction)**
- **Series expansions**
- Root-finding methods (for inverse functions)
- Precomputed approximants (not applicable here)

Brute force: numerical integration

For analytic integrands, there are good algorithms that easily permit achieving 100s or 1000s of digits of accuracy:

- Gaussian quadrature
- Clenshaw-Curtis method (Chebyshev series)
- Trapezoidal rule (for periodic functions)
- Double exponential (tanh-sinh) method
- Taylor series methods (also for ODEs)

Pros:

- Simple, general, flexible approach
- Can deform path of integration as needed

Cons:

- Usually slower than dedicated methods
- Possible convergence problems (oscillation, singularities)
- Error analysis may be complicated for improper integrals

Poisson and the trapezoidal rule (historical remark)

In 1827, Poisson considered the example of the perimeter of an ellipse with axis lengths $1/\pi$ and $0.6/\pi$:

$$I = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - 0.36 \sin^2(\theta)}\, d\theta = \frac{2}{\pi} E(0.36) = 0.9027799 \ldots$$

Poisson used the trapezoidal approximation

$$I \approx I_N = \frac{4}{N} \sum_{k=0}^{N/4}{}' \sqrt{1 - 0.36 \sin^2(2\pi k/N)}.$$

With $N = 16$ (four points!), he computed $I \approx \mathbf{0.9}9\mathbf{27799272}2$ and proved that the error is $< 4.84 \cdot 10^{-6}$.

In fact $|I_N - I| = O(3^{-N})$. See Trefethen & Weideman, *The exponentially convergent trapezoidal rule*, 2014.

A model problem: computing $\exp(x)$

Standard two-step numerical recipe for special functions:
(not all functions fit this pattern, but surprisingly many do!)

1. Argument reduction

$$\exp(x) = \exp(x - n\log(2)) \cdot 2^n$$

$$\exp(x) = \left[\exp(x/2^R)\right]^{2^R}$$

2. Series expansion

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Step (1) ensures rapid convergence and good numerical stability in step (2).

Reducing complexity for p-bit precision

Principles:

- Balance argument reduction and series order optimally
- Exploit special (e.g. hypergeometric) structure of series

How to compute $\exp(x)$ for $x \approx 1$ with an error of 2^{-1000}?

- Only reduction: apply $x \to x/2$ reduction 1000 times
- Only series evaluation: use 170 terms $(170! > 2^{1000})$
- Better: apply $\lceil \sqrt{1000} \rceil = 32$ reductions and use 32 terms

This trick reduces the arithmetic complexity from p to $p^{0.5}$ (time complexity from $p^{2+\varepsilon}$ to $p^{1.5+\varepsilon}$).

With a more complex scheme, the arithmetic complexity can be reduced to $O(\log^2 p)$ (time complexity $p^{1+\varepsilon}$).

Evaluating polynomials using rectangular splitting

(Paterson and Stockmeyer 1973; Smith 1989)

$\sum_{i=0}^{N} \square x^i$ in $O(N)$ cheap steps + $O(N^{1/2})$ expensive steps

$$
\begin{array}{cccccccccc}
(& \square & + & \square x & + & \square x^2 & + & \square x^3 &) & & + \\
(& \square & + & \square x & + & \square x^2 & + & \square x^3 &) & x^4 & + \\
(& \square & + & \square x & + & \square x^2 & + & \square x^3 &) & x^8 & + \\
(& \square & + & \square x & + & \square x^2 & + & \square x^3 &) & x^{12} &
\end{array}
$$

This does not genuinely reduce the asymptotic complexity, but can be a huge improvement (100 times faster) in practice.

Elliptic functions	**Elliptic integrals**

Argument reduction

Move to standard domain (periodicity, modular transformations)	Move parameters close together (various formulas)

Series expansions

Theta function q-series	Multivariate hypergeometric series (Appell, Lauricella ...)

Special cases

Modular forms & functions, theta constants	Complete elliptic integrals, ordinary hypergeometric series (Gauss ${}_2F_1$)

Modular forms and functions

A modular form of weight k is a holomorphic function on $\mathbb{H} = \{\tau : \tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$ satisfying

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k F(\tau)$$

for any integers a, b, c, d with $ad - bc = 1$. A modular function is meromorphic and has weight $k = 0$.

Since $F(\tau) = F(\tau + 1)$, the function has a Fourier series (or Laurent series/ q-expansion)

$$F(\tau) = \sum_{n=-m}^{\infty} c_n e^{2i\pi n\tau} = \sum_{n=-m}^{\infty} c_n q^n, \quad q = e^{2\pi i\tau}, |q| < 1$$

Some useful functions and their q-expansions

Dedekind eta function

- $\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)\sqrt{c\tau+d}\,\eta(\tau)$
- $\eta(\tau) = e^{\pi i\tau/12}\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}$

The j-invariant

- $j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau)$
- $j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$
- $j(\tau) = 32(\theta_2^8+\theta_3^8+\theta_4^8)^3/(\theta_2\theta_3\theta_4)^8$

Theta constants ($q = e^{\pi i\tau}$)

- $(\theta_2,\theta_3,\theta_4) = \sum_{n=-\infty}^{\infty}\left(q^{(n+1/2)^2},\ q^{n^2},\ (-1)^n q^{n^2}\right)$

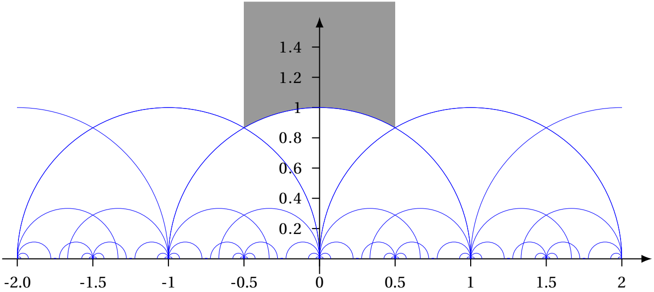
Due to sparseness, we only need $N = O(\sqrt{p})$ terms for p-bit accuracy (so the evaluation takes $p^{1.5+\varepsilon}$ time).

Argument reduction for modular forms

$PSL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

By repeated use of $\tau \to \tau + 1$ or $\tau \to -1/\tau$, we can move τ to the *fundamental domain* $\left\{\tau \in \mathbb{H} : |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}\right\}$.

In the fundamental domain, $|q| \leq \exp(-\pi\sqrt{3}) = 0.00433\ldots$, which gives rapid convergence of the q-expansion.



Practical considerations

Instead of applying $F(\tau + 1) = F(\tau)$ or $F(-1/\tau) = \tau^k F(\tau)$ step by step, build transformation matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and apply to F in one step.

- This improves numerical stability
- g can usually be computed cheaply using machine floats

If computing F via theta constants, apply transformation for F instead of the individual theta constants.

Fast computation of eta and theta function q-series

Consider $\sum_{n=0}^{N} q^{n^2}$. More generally, $q^{P(n)}$, $P \in \mathbb{Z}[x]$ of degree 2.

Naively: $2N$ multiplications.

Enge, Hart & J, *Short addition sequences for theta functions*, 2016:

- Optimized addition sequence for $P(0), P(1), \ldots$ ($2\times$ speedup)
- Rectangular splitting: choose splitting parameter m so that P has few distinct residues mod m (logarithmic speedup, in practice another $2\times$ speedup)

Schost & Nogneng, *On the evaluation of some sparse polynomials*, 2017:

- $N^{1/2+\varepsilon}$ method ($p^{1.25+\varepsilon}$ time complexity) using FFT
- Faster for $p > 200000$ in practice

Jacobi theta functions

Series expansion:

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} w^{2n}, \quad q = e^{\pi i \tau}, w = e^{\pi i z}$$

and similarly for $\theta_1, \theta_2, \theta_4$.

The terms eventually decay rapidly (there can be an initial "hump" if $|w|$ is large). Error bound via geometric series.

For z-derivatives, we compute the object $\theta(z+x,\tau) \in \mathbb{C}[[x]]$ (as a vector of coefficients) in one step.

$$\theta(z+x,\tau) = \theta(z,\tau) + \theta'(z,\tau)x + \ldots + \frac{\theta^{(r-1)}(z,\tau)}{(r-1)!}x^{r-1} + O(x^r) \in \mathbb{C}[[x]]$$

Argument reduction for Jacobi theta functions

Two reductions are necessary:

- Move τ to τ' in the fundamental domain (this operation transforms $z \to z'$, introduces some prefactors, and permutes the theta functions)
- Reduce z' modulo τ' using quasiperiodicity

General formulas for the transformation $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$ are given in (Rademacher, 1973):

$$\theta_n(z, \tau) = \exp(\pi i R/4) \cdot A \cdot B \cdot \theta_S(z', \tau')$$

$$z' = \frac{-z}{c\tau + d}, \quad A = \sqrt{\frac{i}{c\tau + d}}, \quad B = \exp\left(-\pi i c \frac{z^2}{c\tau + d}\right)$$

R, S are integers depending on n and (a, b, c, d).
The argument reduction also applies to $\theta(z + x, \tau) \in \mathbb{C}[[x]]$.

Elliptic functions

The Weierstrass elliptic function $\wp(z,\tau) = \wp(z+1,\tau) = \wp(z+\tau,\tau)$

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0} \left[\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right]$$

is computed via Jacobi theta functions as

$$\wp(z,\tau) = \pi^2\theta_2^2(0,\tau)\theta_3^2(0,\tau)\frac{\theta_4^2(z,\tau)}{\theta_1^2(z,\tau)} - \frac{\pi^2}{3}\left[\theta_3^4(0,\tau) + \theta_3^4(0,\tau)\right]$$

Similarly $\sigma(z,\tau)$, $\zeta(z,\tau)$ and $\wp^{(k)}(z,\tau)$ using z-derivatives of theta functions.

With argument reduction for both z and τ already implemented for theta functions, reduction for $\wp$ is unnecessary (but can improve numerical stability).

Some timings

For d decimal digits ($z = \sqrt{5} + \sqrt{7}i,\ \tau = \sqrt{7} + i/\sqrt{11}$):

Function	$d = 10$	$d = 10^2$	$d = 10^3$	$d = 10^4$	$d = 10^5$
$\exp(z)$	$7.7 \cdot 10^{-7}$	$2.94 \cdot 10^{-6}$	0.000112	0.0062	0.237
$\log(z)$	$8.1 \cdot 10^{-7}$	$2.75 \cdot 10^{-6}$	0.000114	0.0077	0.274
$\eta(\tau)$	$6.2 \cdot 10^{-6}$	$1.99 \cdot 10^{-5}$	0.00037	0.0150	0.69
$j(\tau)$	$6.3 \cdot 10^{-6}$	$2.29 \cdot 10^{-5}$	0.00046	0.0223	1.10
$(\theta_i(0,\tau))_{i=1}^4$	$7.6 \cdot 10^{-6}$	$2.67 \cdot 10^{-5}$	0.00044	0.0217	1.09
$(\theta_i(z,\tau))_{i=1}^4$	$2.8 \cdot 10^{-5}$	$8.10 \cdot 10^{-5}$	0.00161	0.0890	5.41
$\wp(z,\tau)$	$3.9 \cdot 10^{-5}$	0.000122	0.00213	0.113	6.55
$(\wp, \wp')$	$5.6 \cdot 10^{-5}$	0.000166	0.00255	0.128	7.26
$\zeta(z,\tau)$	$7.5 \cdot 10^{-5}$	0.000219	0.00284	0.136	7.80
$\sigma(z,\tau)$	$7.6 \cdot 10^{-5}$	0.000223	0.00299	0.143	8.06

Elliptic integrals

Any elliptic integral $\int R(x, \sqrt{P(x)})\,dx$ can be written in terms of a small "basis set". The *Legendre forms* are used by tradition.

Complete elliptic integrals:

$$K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-m\sin^2 t}} = \int_0^1 \frac{dt}{\left(\sqrt{1-t^2}\right)\left(\sqrt{1-mt^2}\right)}$$

$$E(m) = \int_0^{\pi 2} \sqrt{1 - m\sin^2 t}\,dt = \int_0^1 \frac{\sqrt{1-mt^2}}{\sqrt{1-t^2}}\,dt$$

$$\Pi(n, m) = \int_0^{\pi/2} \frac{dt}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}} = \int_0^1 \frac{dt}{(1-nt^2)\sqrt{1-t^2}\sqrt{1-mt^2}}$$

Incomplete integrals:

$$F(\phi, m) = \int_0^{\phi} \frac{dt}{\sqrt{1-m\sin^2 t}} = \int_0^{\sin\phi} \frac{dt}{\left(\sqrt{1-t^2}\right)\left(\sqrt{1-mt^2}\right)}$$

$$E(\phi, m) = \int_0^{\phi} \sqrt{1 - m\sin^2 t}\,dt = \int_0^{\sin\phi} \frac{\sqrt{1-mt^2}}{\sqrt{1-t^2}}\,dt$$

$$\Pi(n, \phi, m) = \int_0^{\phi} \frac{dt}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}} = \int_0^{\sin\phi} \frac{dt}{(1-nt^2)\sqrt{1-t^2}\sqrt{1-mt^2}}$$

Complete elliptic integrals and ${}_2F_1$

The Gauss hypergeometric function is defined for $|z| < 1$ by

$$ {}_2F_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (x)_k = x(x+1)\cdots(x+k-1) $$

and elsewhere by analytic continuation. The ${}_2F_1$ function can be computed efficiently for any $z \in \mathbb{C}$.

$$ K(m) = \tfrac{1}{2}\pi \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1, m) $$

$$ E(m) = \tfrac{1}{2}\pi \, {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}, 1, m) $$

This works, but it's not the best way!

Complete elliptic integrals and the AGM

The AGM of x, y is the common limit of the sequences

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}$$

with $a_0 = x, b_0 = y$. As a functional equation:

$$M(x,y) = M\left(\frac{x+y}{2}, \sqrt{xy}\right)$$

Each step *doubles the number of digits* in $M(x,y) \approx x \approx y$
$\Rightarrow$ convergence in $O(\log p)$ operations ($p^{1+\varepsilon}$ time complexity).

$$K(m) = \frac{\pi}{2M(1, \sqrt{1-m})}, \quad E(m) = (1-m)(2mK'(m) + K(m))$$

Numerical aspects of the AGM

Argument reduction vs series expansion: $O(1)$ terms only. Slightly better than reducing all the way to $|a_n - b_n| < 2^{-p}$:

$$\frac{\pi}{4K(z^2)} = \frac{1}{2} - \frac{z^2}{8} - \frac{5z^4}{128} - \frac{11z^6}{512} - \frac{469z^8}{32768} + O(z^{10})$$

Complex variables: simplify to $M(z) = M(1, z)$ using $M(x, y) = xM(1, y/x)$. Some case distinctions for correct square root branches in AGM iteration.

Derivatives: can use finite (central) difference for $M'(z)$ (better method possible using elliptic integrals), higher derivatives using recurrence relations.

Incomplete elliptic integrals

Incomplete elliptic integrals are multivariate hypergeometric functions. In terms of the Appell F_1 function

$$F_1(a, b_1, b_2; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}\, m!\, n!}\, x^m y^n$$

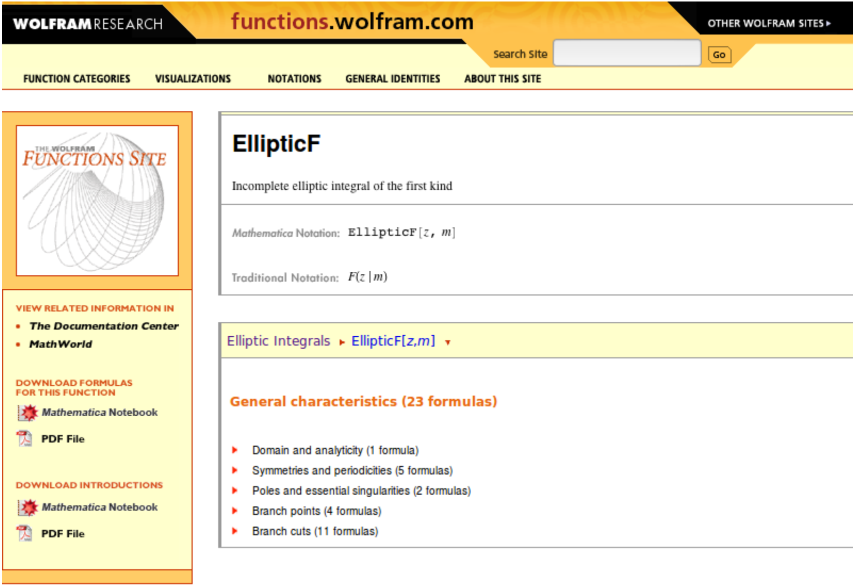
where $|x|, |y| < 1$, we have

$$F(z, m) = \int_0^z \frac{dt}{\sqrt{1 - m \sin^2 t}} = \sin(z)\, F_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; \sin^2 z, m \sin^2 z)$$

Problems:

- How to reduce arguments so that $|x|, |y| \ll 1$?
- How to perform analytic continuation and obtain consistent branch cuts for complex variables?

Branch cuts of Legendre incomplete elliptic integrals



Branch cuts of $F(z, m)$ with respect to z . . .

Elliptic Integrals ▸ EllipticF[z,m] ▸ General characteristics ▸ Branch cuts ▾

With respect to z

General description

For fixed m, the function $F(z \mid m)$ can have up to six infinite sets of branch cuts (it has at least four), which form very complicated curves in the case of generic m.

For fixed real $m < 1$, the function $F(z \mid m)$ does not have branch cuts on the real axis and on the vertical intervals $\{\csc^{-1}(\sqrt{m}) + \pi k, \pi - \csc^{-1}(\sqrt{m}) + \pi k\}$ /; $k \in \mathbb{Z} \land m \in (-\infty, 1)$.

For fixed real $m < 1$, the function $F(z \mid m)$ has four infinite sets of branch cuts located on vertical intervals starting at the points $z == \pi k \pm \csc^{-1}(\sqrt{m})$ /; $k \in \mathbb{Z}$ and extending to imaginary infinity.

For fixed generic m, the function $F(z \mid m)$ has the following six infinite sets of branch cuts:

1) real intervals $\left\{\pi k + \csc^{-1}(\sqrt{m}), \pi k + \frac{\pi}{2}\right\}$ /; $k \in \mathbb{Z} \land m > 1$, where $F(z \mid m)$ is continuous from below (for generic complex m, these branch cuts deform into complicated curves); in the case $m < 1$ these real intervals vanish

2) real intervals $\left\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{m})\right\}$ /; $k \in \mathbb{Z} \land m > 1$, where $F(z \mid m)$ is continuous from above (for generic complex m, these branch cuts deform into complicated curves); in the case $m < 1$ these real intervals vanish

3) vertical intervals $\left\{\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\right\}$ /; $k \in \mathbb{Z} \land m \notin (0, 1)$, or $\left\{\pi - \csc^{-1}(\sqrt{m}) + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\right\}$ /; $k \in \mathbb{Z} \land m \in (0, 1)$, where $F(z \mid m)$ is continuous from the left

4) vertical intervals $\left\{\frac{3\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k + i\infty\right\}$ /; $k \in \mathbb{Z} \land m \notin (0, 1)$, or

Branch cuts of $F(z, m)$ with respect to z (continued)

$\left\{2\pi - \csc^{-1}(\sqrt{m}) + 2\pi k,\ \frac{3\pi}{2} + 2\pi k + i\infty\right\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $F(z \mid m)$ is continuous from the right

5) vertical intervals $\left\{\frac{\pi}{2} + 2\pi k - i\infty,\ \frac{\pi}{2} + 2\pi k\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)$, or

$\left\{\frac{\pi}{2} + 2\pi k - i\infty, 2\pi k + \csc^{-1}(\sqrt{m})\right\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $F(z \mid m)$ is continuous from the left

6) vertical intervals $\left\{\frac{3\pi}{2} + 2\pi k - i\infty,\ \frac{3\pi}{2} + 2\pi k\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)$, or

$\left\{\frac{3\pi}{2} + 2\pi k - i\infty, 2\pi k + \pi + \csc^{-1}(\sqrt{m})\right\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $F(z \mid m)$ is continuous from the right.

$$
\begin{aligned}
\mathcal{BC}_z(F(z \mid m)) = \Big\{ & \left\{\left\{\left(\pi k + \csc^{-1}(\sqrt{m}), \pi k + \frac{\pi}{2}\right), i\right\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land m > 1\right\}, \\
& \left\{\left\{\left(\pi k + \frac{\pi}{2}, \pi(k+1) - \csc^{-1}(\sqrt{m})\right), -i\right\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land m > 1\right\}, \\
& \left\{\left\{\left(2\pi k + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2} + i\infty\right), 1\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)\right\} \bigvee \\
& \quad \left\{\left\{\left(2\pi k + \pi - \csc^{-1}(\sqrt{m}), 2k\pi + \frac{\pi}{2} + i\infty\right), 1\right\} /; k \in \mathbb{Z} \land m \in (0, 1)\right\}, \\
& \left\{\left\{\left(2\pi k + \frac{3\pi}{2}, 2k\pi + \frac{3\pi}{2} + i\infty\right), -1\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)\right\} \bigvee \\
& \quad \left\{\left\{\left(2\pi k + 2\pi - \csc^{-1}(\sqrt{m}), 2k\pi + \frac{3\pi}{2} + i\infty\right), -1\right\} /; k \in \mathbb{Z} \land m \in (0, 1)\right\}, \\
& \left\{\left\{\left(2\pi k + \frac{\pi}{2} - i\infty, 2k\pi + \frac{\pi}{2}\right), 1\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)\right\} \bigvee \\
& \quad \left\{\left\{\left(2\pi k + \frac{\pi}{2} - i\infty, 2\pi k + \csc^{-1}(\sqrt{m})\right), 1\right\} /; k \in \mathbb{Z} \land m \in (0, 1)\right\}, \\
& \left\{\left\{\left(2\pi k + \frac{\pi}{2} - i\infty, 2k\pi + \frac{\pi}{2}\right), -1\right\} /; k \in \mathbb{Z} \land m \notin (0, 1)\right\} \bigvee \\
& \quad \left\{\left\{\left(2\pi k + \frac{\pi}{2} - i\infty, 2\pi k + \pi + \csc^{-1}(\sqrt{m})\right), -1\right\} /; k \in \mathbb{Z} \land m \in (0, 1)\right\}
\end{aligned}
$$

Branch cuts of $F(z, m)$ with respect to z (continued)

Formulas on real axis for real *m*

For $m<1$

For fixed real $m < 1$, the function $F(z \mid m)$ does not have branch cuts on the real axis.

For $m>1$

$$\lim_{\epsilon \to +0} F(x + i\epsilon \mid m) = -F(z \mid m) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4\left(\left\lfloor \frac{z}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m) \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \csc^{-1}\left(\sqrt{m}\right) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

$$\lim_{\epsilon \to +0} F(x - i\epsilon \mid m) = F(x \mid m) \;/;\; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \csc^{-1}\left(\sqrt{m}\right) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

$$\lim_{\epsilon \to +0} F(x + i\epsilon \mid m) = F(x \mid m) \;/;\; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \frac{\pi}{2} + \pi k < x < \pi(k+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

$$\lim_{\epsilon \to +0} F(x - i\epsilon \mid m) = -F(x \mid m) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m) \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \frac{\pi}{2} < x < \pi(k+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

Branch cuts of $F(z, m)$ with respect to z (continued)

Formulas for vertical intervals

For $m<1$

For fixed real $m<1$, the function $F(z\,|\,m)$ has branch points $\csc^{-1}\left(\sqrt{m}\right)+\pi k$ /; $k\in\mathbb{Z}$ and $\pi-\csc^{-1}\left(\sqrt{m}\right)+\pi k$ /; $k\in\mathbb{Z}$. In this case branch cuts lay at the vertical lines beginning from these points and going to imaginary infinity. By this reason for fixed real $m<1$, the function $F(z\,|\,m)$ does not have branch cuts on the vertical intervals

For $m>0$

$$\lim_{\epsilon\to+0} F\left(2\pi k+i x+\frac{\pi}{2}-\epsilon\,\middle|\,m\right)=F\left(2\pi k+i x+\frac{\pi}{2}\,\middle|\,m\right) \text{ /; } x\in\mathbb{R}\wedge k\in\mathbb{Z}$$

$$\lim_{\epsilon\to+0} F\left(2\pi k+i x+\frac{\pi}{2}+\epsilon\,\middle|\,m\right)=-F\left(i x+\frac{\pi}{2}\,\middle|\,m\right)-\frac{2}{\sqrt{m}}K\left(\frac{1}{m}\right)+4(k+1)K(m) \text{ /; }$$
$$m\in\mathbb{R}\bigwedge x\in\mathbb{R}\bigwedge\left(0<m<1\bigwedge x>-\operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right)\right)\bigvee(m>1\wedge x<0)\wedge k\in\mathbb{Z}$$

$$\lim_{\epsilon\to+0} F\left(2\pi k+i x+\frac{\pi}{2}+\epsilon\,\middle|\,m\right)=-F\left(i x+\frac{\pi}{2}\,\middle|\,m\right)+\frac{2}{\sqrt{m}}K\left(\frac{1}{m}\right)+4kK(m) \text{ /; }$$
$$m\in\mathbb{R}\bigwedge x\in\mathbb{R}\bigwedge\left(0<m<1\bigwedge x<\operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right)\bigvee m>1\wedge x>0\right)\bigwedge k\in\mathbb{Z}$$

Branch cuts of $F(z, m)$ with respect to z (continued)

- $$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} - \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{3\pi}{2} \,\middle|\, m\right) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4(k+2) K(m) /;$$
  $$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x > -\operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right) \bigvee m > 1 \wedge x < 0\right) \bigwedge k \in \mathbb{Z}$$

- $$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} - \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{3\pi}{2} \,\middle|\, m\right) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4(k+1) K(m) /;$$
  $$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x < \operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right) \bigvee m > 1 \wedge x > 0\right) \bigwedge k \in \mathbb{Z}$$

- $$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} + \epsilon \,\middle|\, m\right) = F\left(2\pi k + i x + \frac{3\pi}{2} \,\middle|\, m\right) /; x \in \mathbb{R} \wedge k \in \mathbb{Z}$$

Branch cuts of $F(z, m)$ with respect to m

EllipticF

Incomplete elliptic integral of the first kind

Mathematica Notation: `EllipticF[z, m]`

Traditional Notation: $F(z \mid m)$

Elliptic Integrals ▸ EllipticF[z,m] ▸ General characteristics ▸ Branch cuts ▾

With respect to *m* (0 formulas)

Branch cut locations: complicated.

Conclusion: the Legendre forms are not nice as building blocks.

Carlson's symmetric forms

In the 1960s, Bille C. Carlson suggested an alternative "basis set" for incomplete elliptic integrals:

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}}$$

$$R_C(x,y) = R_F(x,y,y), \quad R_D(x,y,z) = R_J(x,y,z,z)$$

Advantages:

- Symmetry unifies and simplifies transformation laws
- Symmetry greatly simplifies series expansions
- The functions have nice complex branch structure
- Simple universal algorithm for computation

Evaluation of Legendre forms

For $-\frac{\pi}{2} \leq \mathrm{Re}(z) \leq \frac{\pi}{2}$:

$$F(z, m) = \sin(z)\, R_F(\cos^2(z), 1 - m\sin^2(z), 1)$$

Elsewhere, use quasiperiodic extension:

$$F(z + k\pi, m) = 2kK(m) + F(z, m), \quad k \in \mathbb{Z}$$

Similarly for $E(z, m)$ and $\Pi(n, z, m)$.

Slight complication to handle (complex) intervals straddling the lines $\mathrm{Re}(z) = (n + \frac{1}{2})\pi$.

Useful for implementations: variants with $z \to \pi z$.

Symmetric argument reduction

We have the functional equation

$$R_F(x,y,z) = R_F\left(\frac{x+\lambda}{4}, \frac{y+\lambda}{4}, \frac{z+\lambda}{4}\right)$$

where $\lambda = \sqrt{x}\sqrt{y} + \sqrt{y}\sqrt{z} + \sqrt{z}\sqrt{x}$. Each application reduces the distance between x, y, z by a factor $1/4$.

Algorithm: apply reduction until the distance is ε, then use an order-N series expansion with error term $O(\varepsilon^N)$.

For p-bit accuracy, need $p/(2N)$ argument reduction steps.

(A similar functional equation exists for $R_J(x,y,z,p)$.)

Series expansion when arguments are close

$$R_F(x,y,z) = R_{-1/2}\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},x,y,z\right)$$

$$R_J(x,y,z,p) = R_{-3/2}\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},x,y,z,p,p\right)$$

Carlson's R is a multivariate hypergeometric series:

$$\begin{aligned} R_{-a}(\mathbf{b};\mathbf{z}) &= \sum_{M=0}^{\infty} \frac{(a)_M}{(\sum_{j=1}^{n} b_j)_M} T_M(b_1,\ldots,b_n; 1-z_1,\ldots,1-z_n) \\ &= \sum_{M=0}^{\infty} \frac{z_n^{-a}\,(a)_M}{(\sum_{j=1}^{n} b_j)_M} T_M\!\left(b_1,\ldots,b_{n-1}; 1-\frac{z_1}{z_n},\ldots,1-\frac{z_{n-1}}{z_n}\right), \end{aligned}$$

$$T_M(b_1,\ldots,b_n,w_1,\ldots,w_n) = \sum_{m_1+\ldots+m_n=M} \prod_{j=1}^{n} \frac{(b_j)_{m_j}}{(m_j)!} w_j^{m_j}$$

Note that $|T_M| \le Const \cdot p(M) \max(|w_1|,\ldots,|w_n|)^M$, so we can easily bound the tail by a geometric series.

A clever idea by Carlson: symmetric polynomials

Using elementary symmetric polynomials $E_s(w_1, \ldots, w_n)$,

$$T_M(\tfrac{1}{2}, \mathbf{w}) = \sum_{m_1+2m_2+\ldots+nm_n=M} (-1)^{M+\sum_j m_j} \left(\tfrac{1}{2}\right)_{\sum_j m_j} \prod_{j=1}^{n} \frac{E_j^{m_j}(\mathbf{w})}{(m_j)!}$$

We can expand R around the mean of the arguments, taking $w_j = 1 - z_j/A$ where $A = \frac{1}{n}\sum_{j=1}^{n} z_j$. Then $E_1 = 0$, and most of the terms disappear!

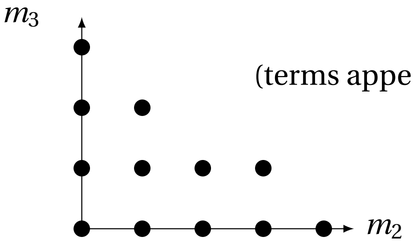
Carlson suggested expanding to $M < N = 8$:

$$A^{1/2} R_F(x, y, z) = 1 - \frac{E_2}{10} + \frac{E_3}{14} + \frac{E_2^2}{24} - \frac{3E_2E_3}{44} - \frac{5E_2^3}{208} + \frac{3E_3^2}{104} + \frac{E_2^2E_3}{16} + O(\varepsilon^8)$$

Need $p/16$ argument reduction steps for p-bit accuracy.

Rectangular splitting for the R series

The exponents of $E_2^{m_2} E_3^{m_3}$ appearing in the series for R_F are the lattice points $m_2, m_3 \in \mathbb{Z}_{\geq 0}$ with $2m_2 + 3m_3 < N$.



(terms appearing for $N = 10$)

Compute powers of E_2, use Horner's rule with respect to E_3.
Clear denominators so that all coefficients are small integers.
$\Rightarrow$ $O(N^2)$ cheap steps + $O(N)$ expensive steps

For R_J, compute powers of E_2, E_3, use Horner for E_4, E_5.

Balancing series evaluation and argument reduction

Consider R_F:

p = wanted precision in bits
$O(\varepsilon^N)$ = error due to truncating the series expansion
$O(N^2)$ = number of terms in series
$O(p/N)$ = number of argument reduction steps for $\varepsilon^N = 2^{-p}$

Overall cost $O(N^2 + p/N)$ is minimized by $N \sim p^{0.333}$, giving $p^{0.667}$ arithmetic complexity ($p^{1.667}$ time complexity).

Empirically, $N \approx 2p^{0.4}$ is optimal (due to rectangular splitting).
Speedup over $N = 8$ at d digits precision:

$d = 10$	$d = 10^2$	$d = 10^3$	$d = 10^4$	$d = 10^5$
1	1.5	4	11	31

Some timings

We include $K(m)$ (computed by AGM), $F(z,m)$ (computed by R_F) and the inverse Weierstrass elliptic function:

$$\wp^{-1}(z,\tau) = \frac{1}{2}\int_z^\infty \frac{dt}{\sqrt{(t-e_1)(t-e_2)(t-e_3)}} = R_F(z-e_1, z-e_2, z-e_3)$$

Function	$d = 10$	$d = 10^2$	$d = 10^3$	$d = 10^4$	$d = 10^5$
$\exp(z)$	$7.7 \cdot 10^{-7}$	$2.94 \cdot 10^{-6}$	0.000112	0.0062	0.237
$\log(z)$	$8.1 \cdot 10^{-7}$	$2.75 \cdot 10^{-6}$	0.000114	0.0077	0.274
$\eta(\tau)$	$6.2 \cdot 10^{-6}$	$1.99 \cdot 10^{-5}$	0.00037	0.0150	0.693
$K(m)$	$5.4 \cdot 10^{-6}$	$1.97 \cdot 10^{-5}$	0.000182	0.0068	0.213
$F(z,m)$	$2.4 \cdot 10^{-5}$	0.000114	0.0022	0.187	19.1
$\wp(z,\tau)$	$3.9 \cdot 10^{-5}$	0.000122	0.00214	0.129	6.82
$\wp^{-1}(z,\tau)$	$3.1 \cdot 10^{-5}$	0.000142	0.00253	0.202	19.7

Quadratic transformations

It is possible to construct AGM-like methods (converging in $O(\log p)$ steps) for general elliptic integrals and functions.

Problems:

- The overhead may be slightly higher at low precision
- Correct treatment of complex variables is not obvious

Unfortunately, I have not had time to study this topic. However, see the following papers:

- The elliptic logarithm ($\approx \wp^{-1}$): John E. Cremona and Thotsaphon Thongjunthug, *The complex AGM, periods of elliptic curves over and complex elliptic logarithms*, 2013.
- Elliptic and theta functions: Hugo Labrande, *Computing Jacobi's θ in quasi-linear time*, 2015.