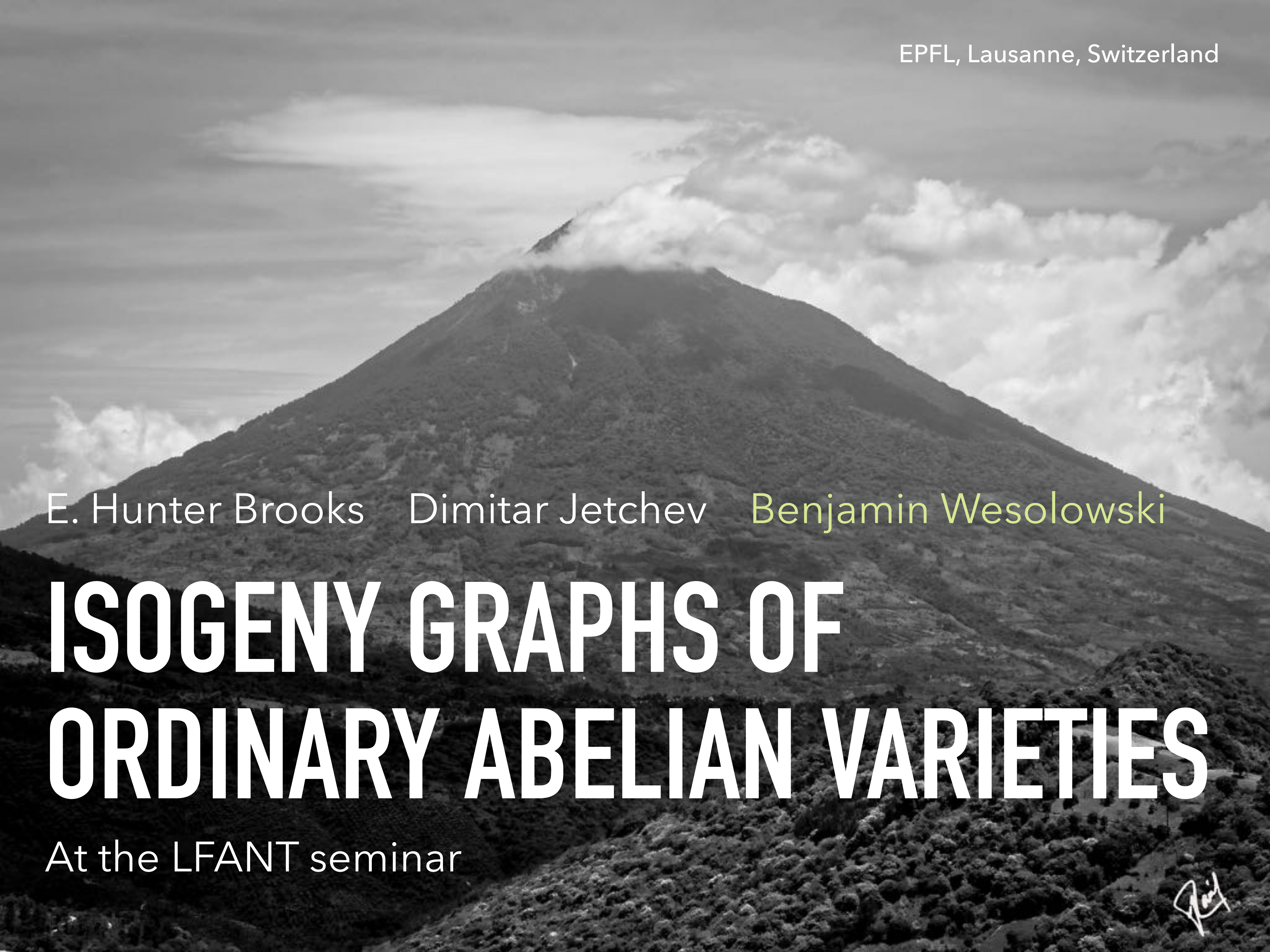


EPFL, Lausanne, Switzerland

E. Hunter Brooks   Dimitar Jetchev   Benjamin Wesolowski

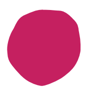
ISOGENY GRAPHS OF ORDINARY ABELIAN VARIETIES

At the LFANT seminar

AN NON-VIOLENT INTRODUCTION TO

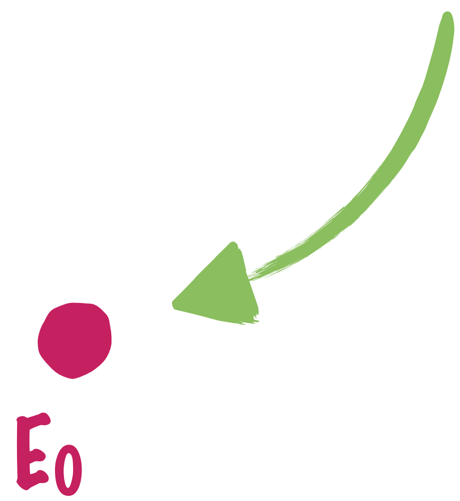
ISOGENY GRAPHS

ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES



ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES

This vertex represents an elliptic curve E_0 over a finite field F



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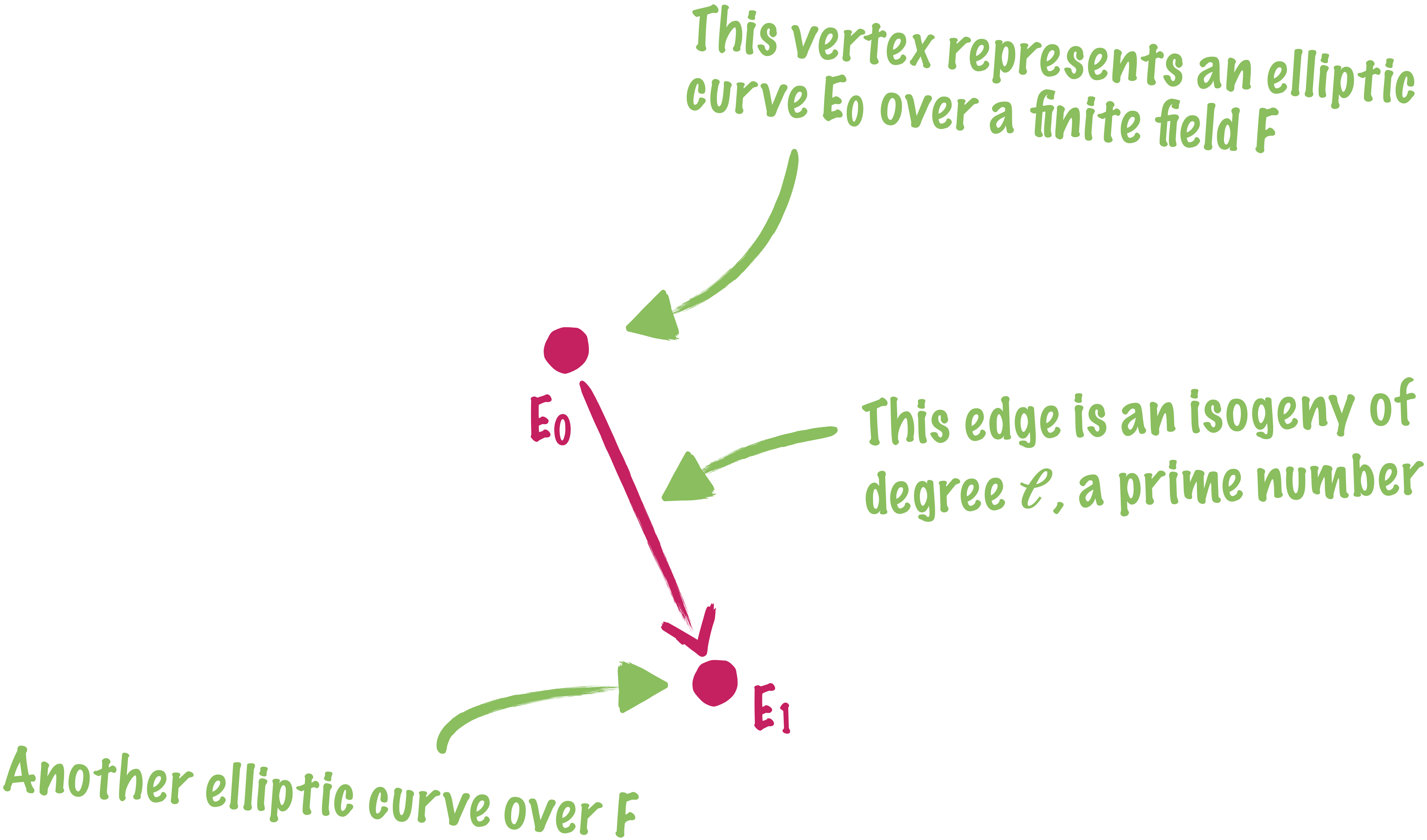
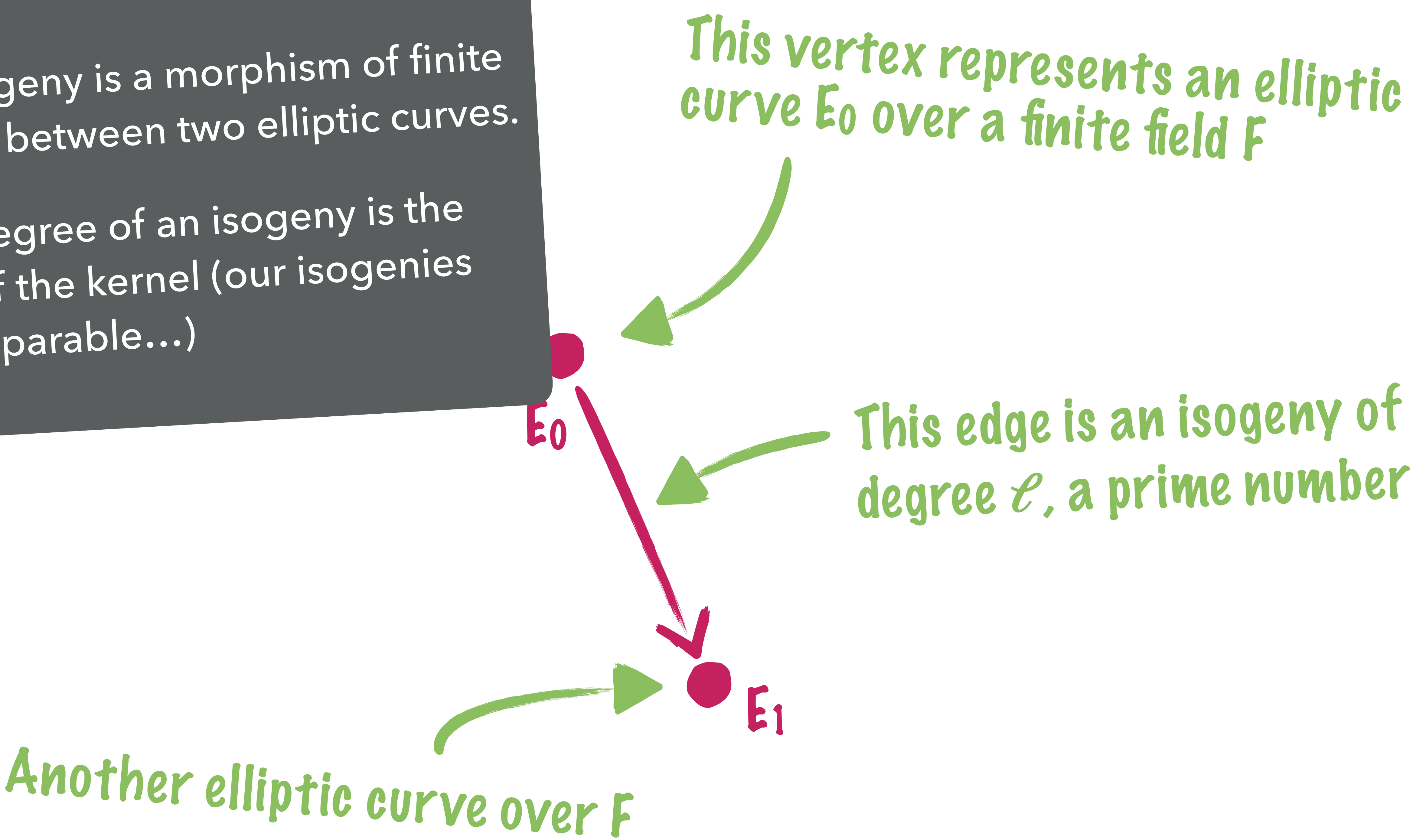


ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES

An isogeny is a morphism of finite kernel between two elliptic curves.

The degree of an isogeny is the size of the kernel (our isogenies are separable…)



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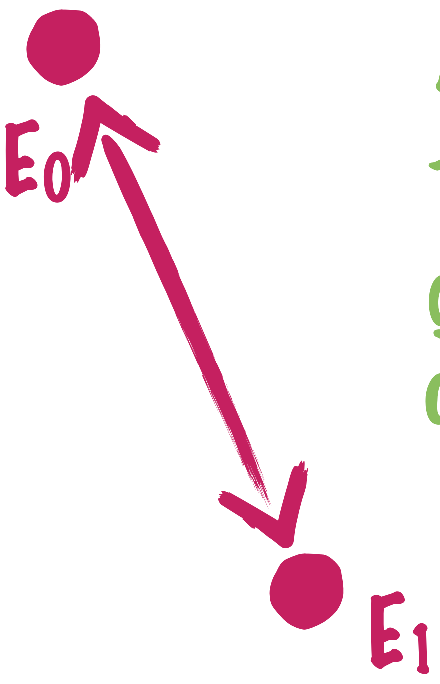

Any isogeny has a dual of the same degree (here, ℓ) going in the opposite direction

ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES

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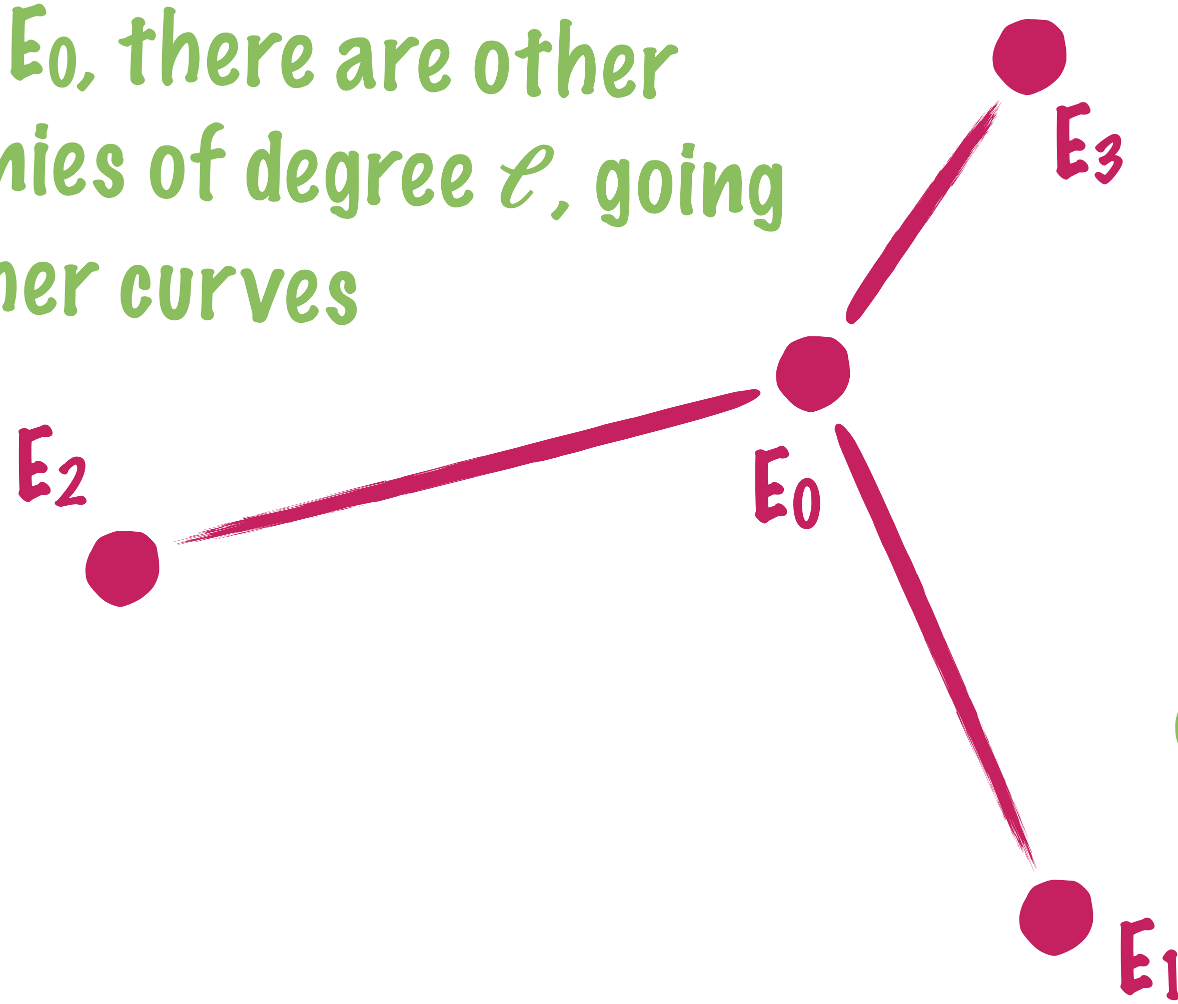
So we represent it by a non-directed edge

ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES

From E_0, there are other isogenies of degree ℓ, going to other curves



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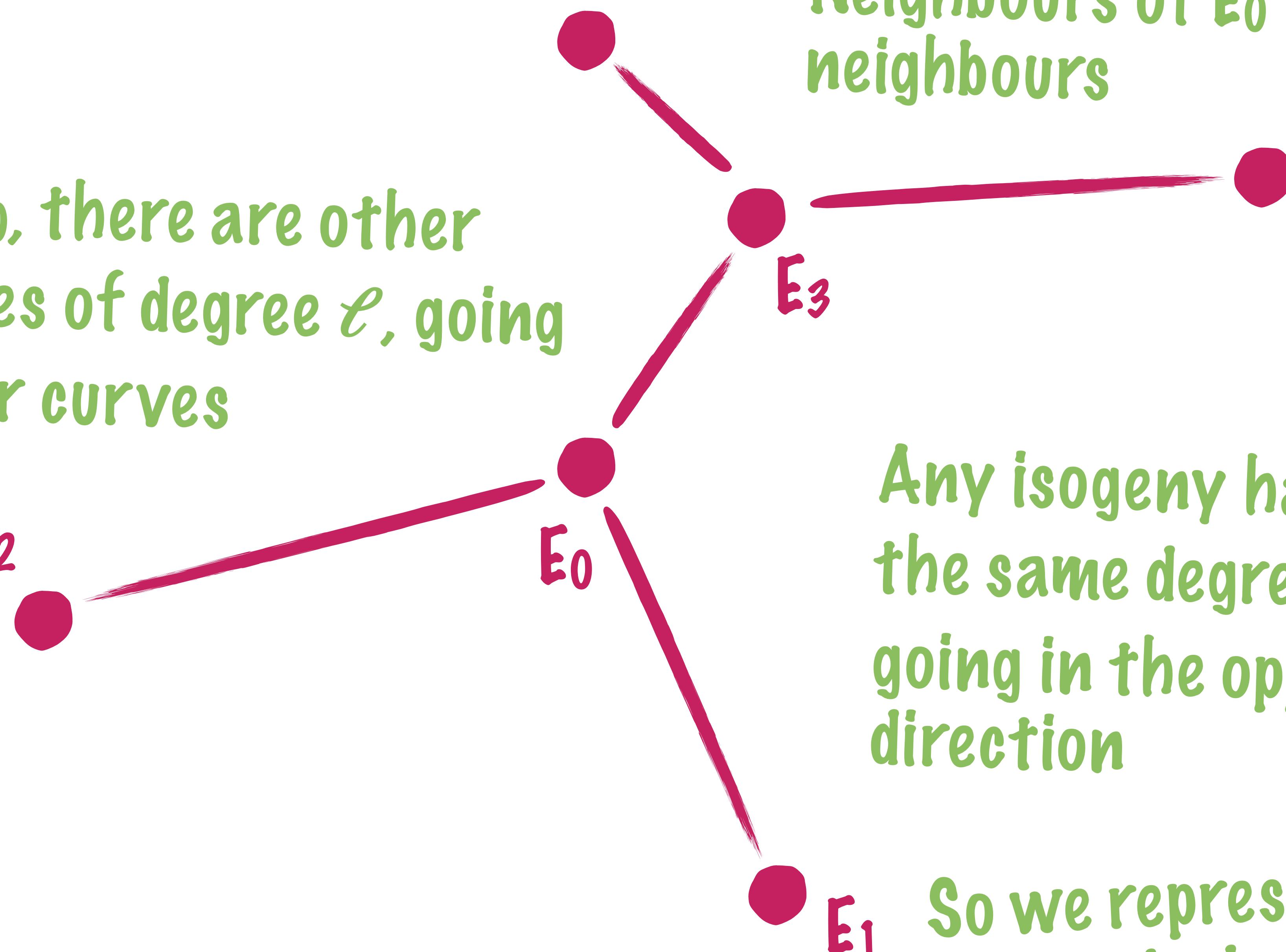
ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES

Neighbours of E_0 have more neighbours

From E_0, there are other isogenies of degree ℓ, going to other curves



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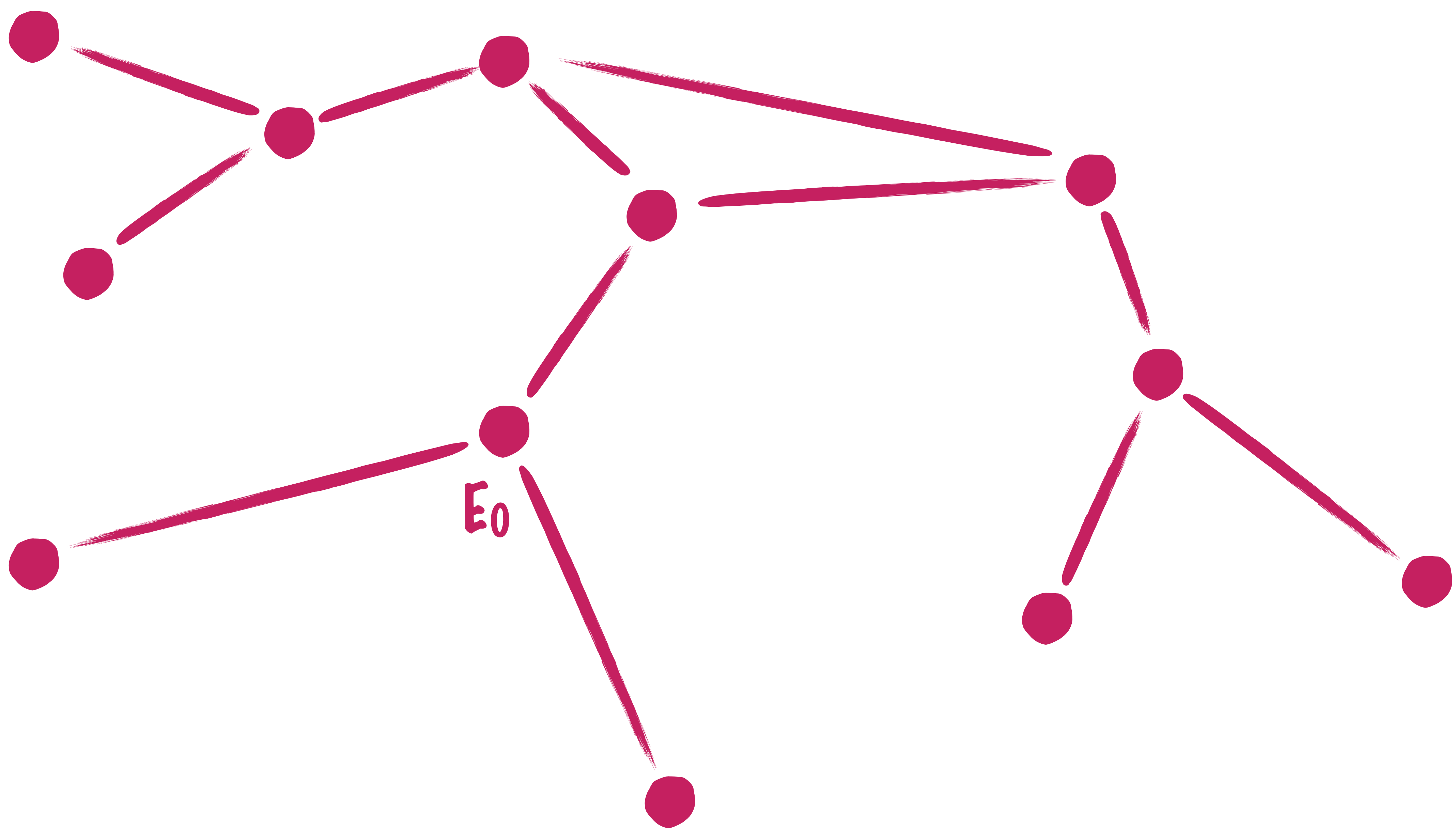
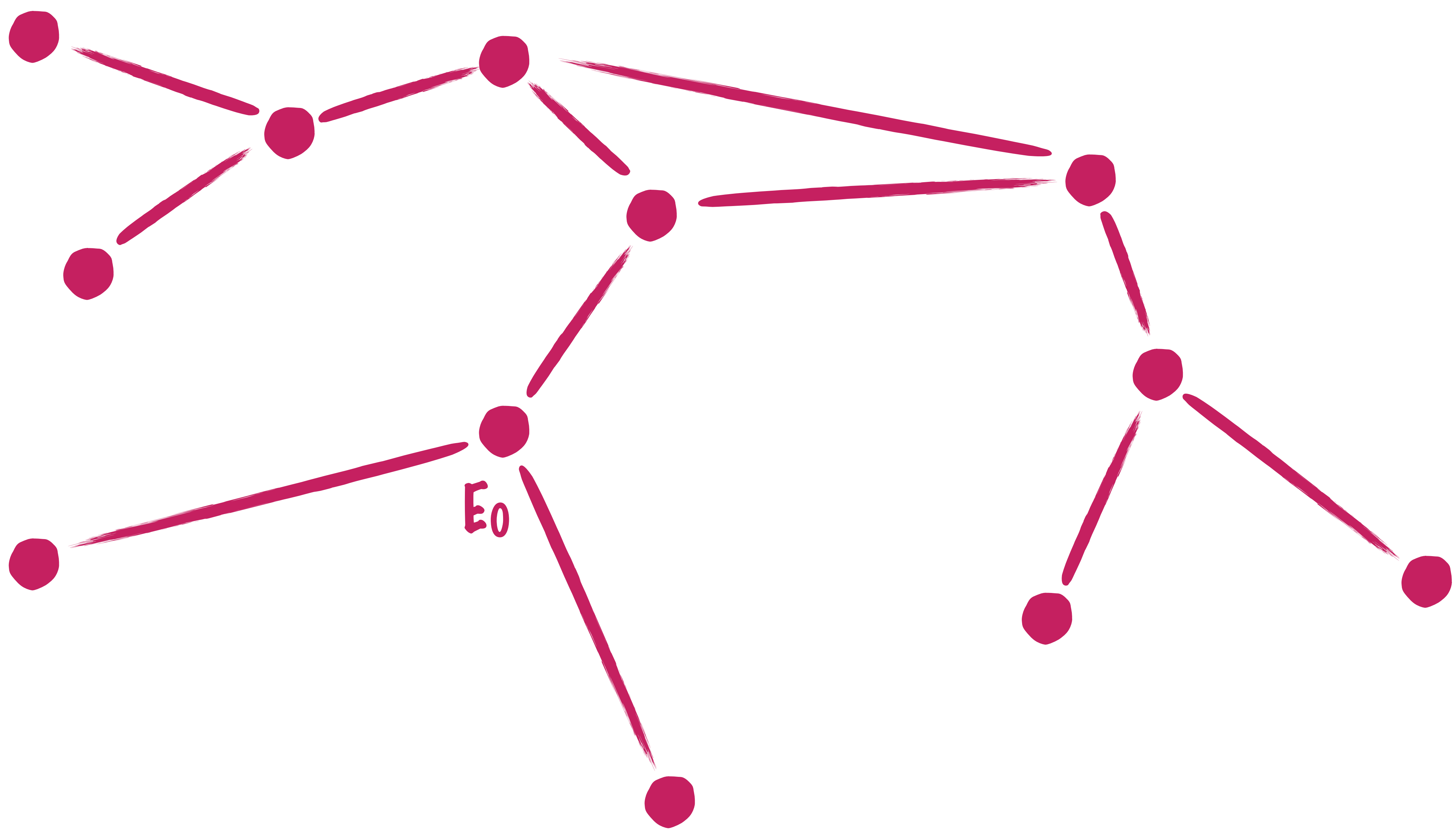


Once all the possible neighbours have been reached, we obtain the connected graph of ℓ-isogenies of E_0

ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES



This one is a typical example!

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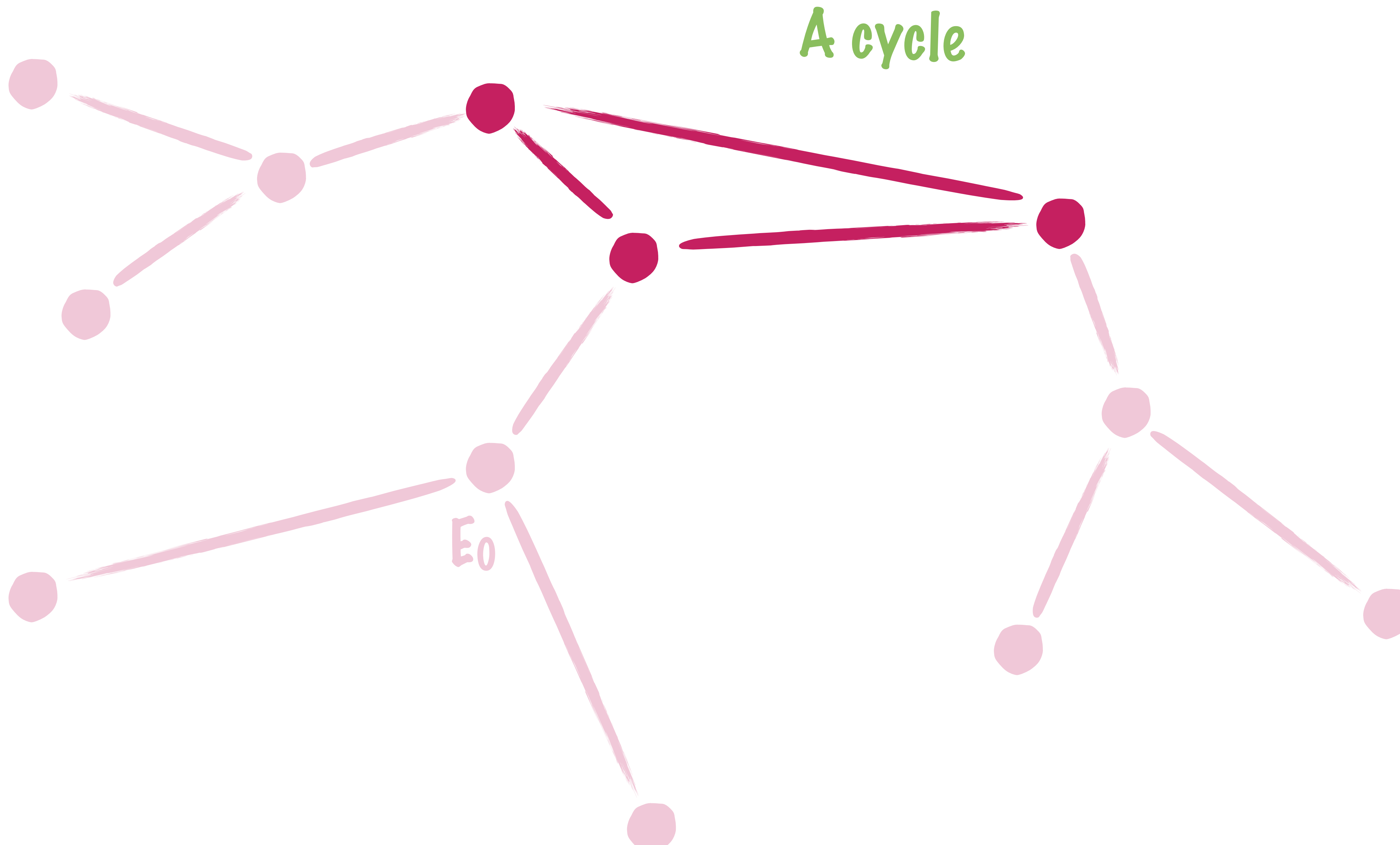

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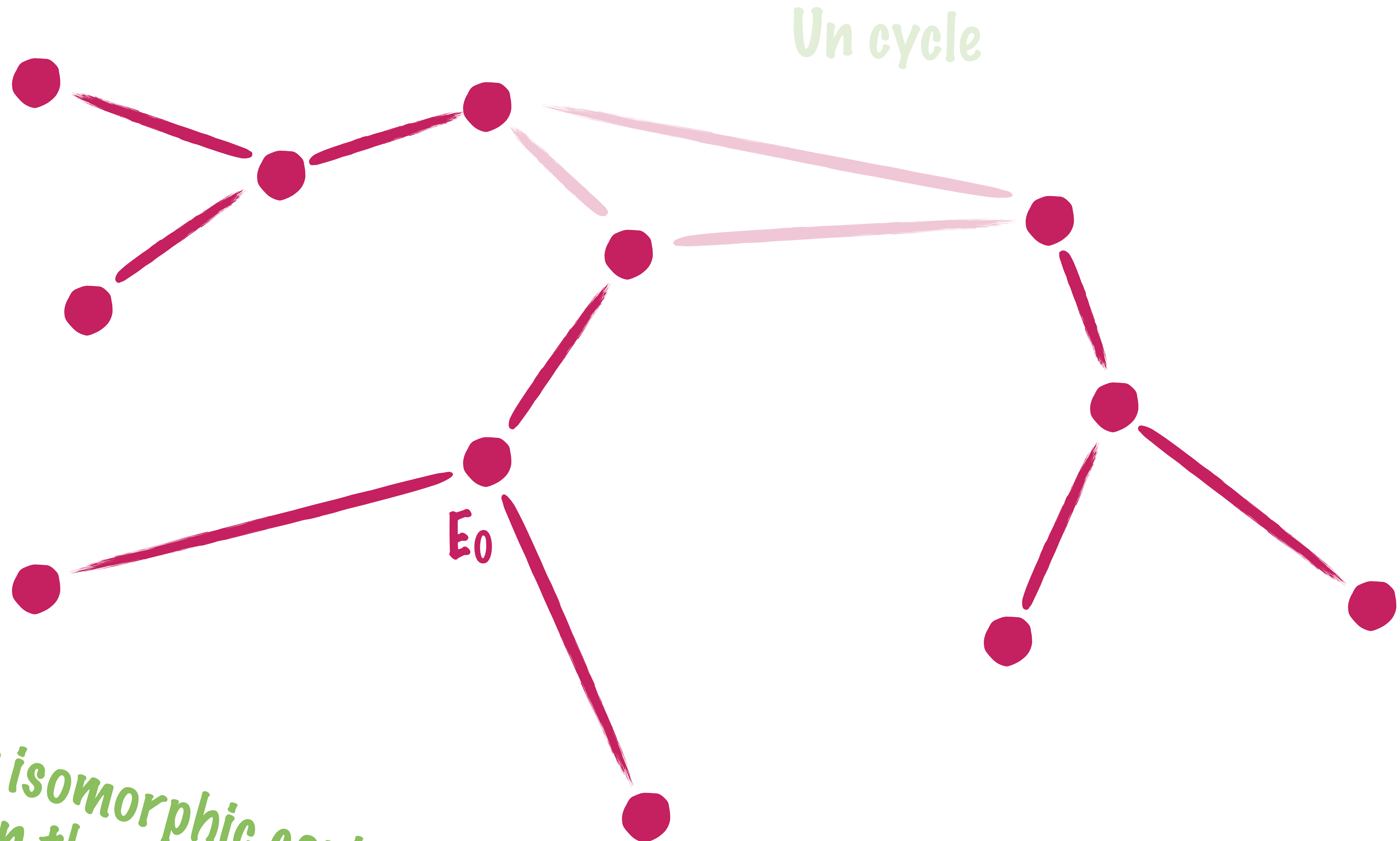
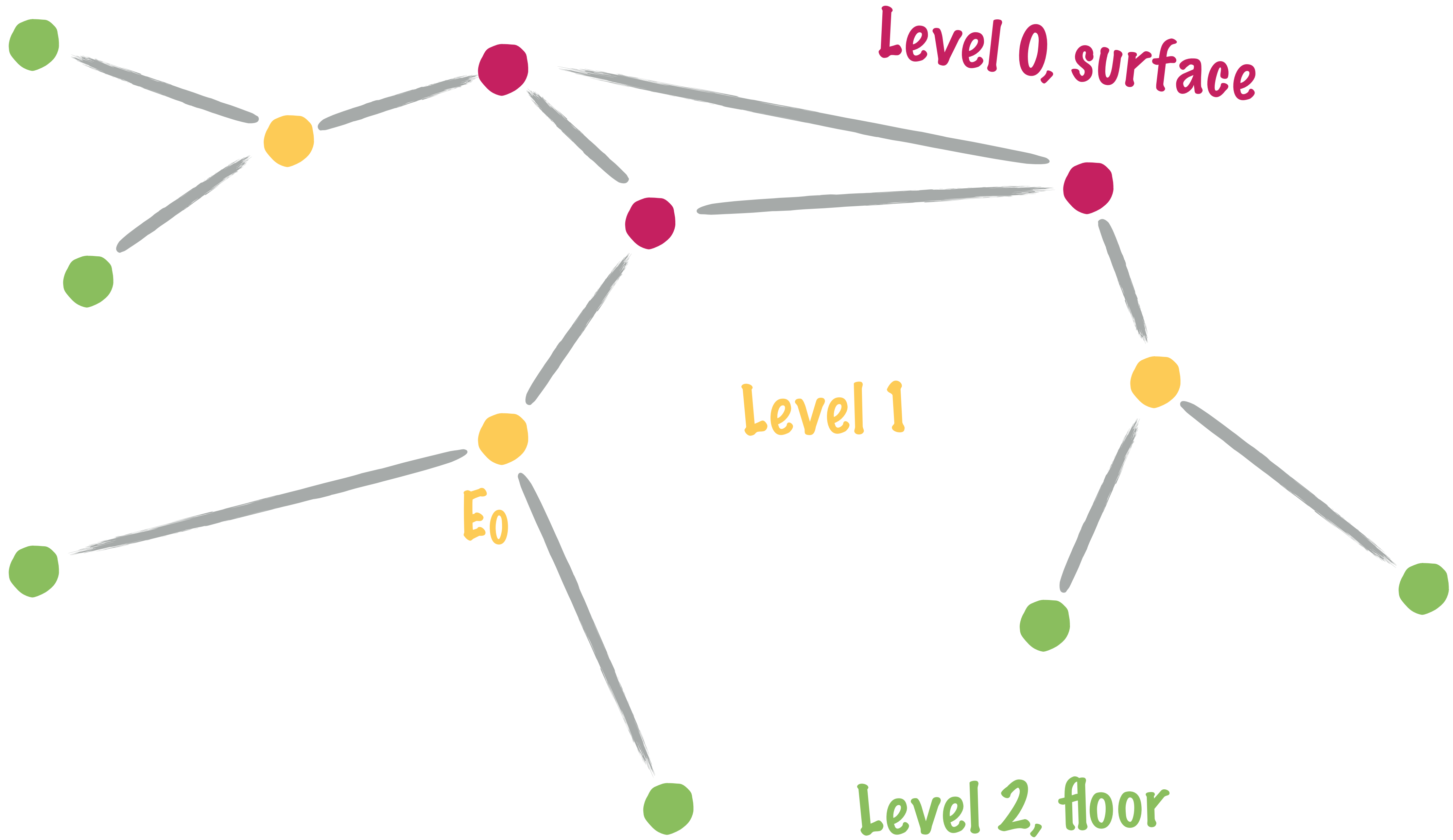
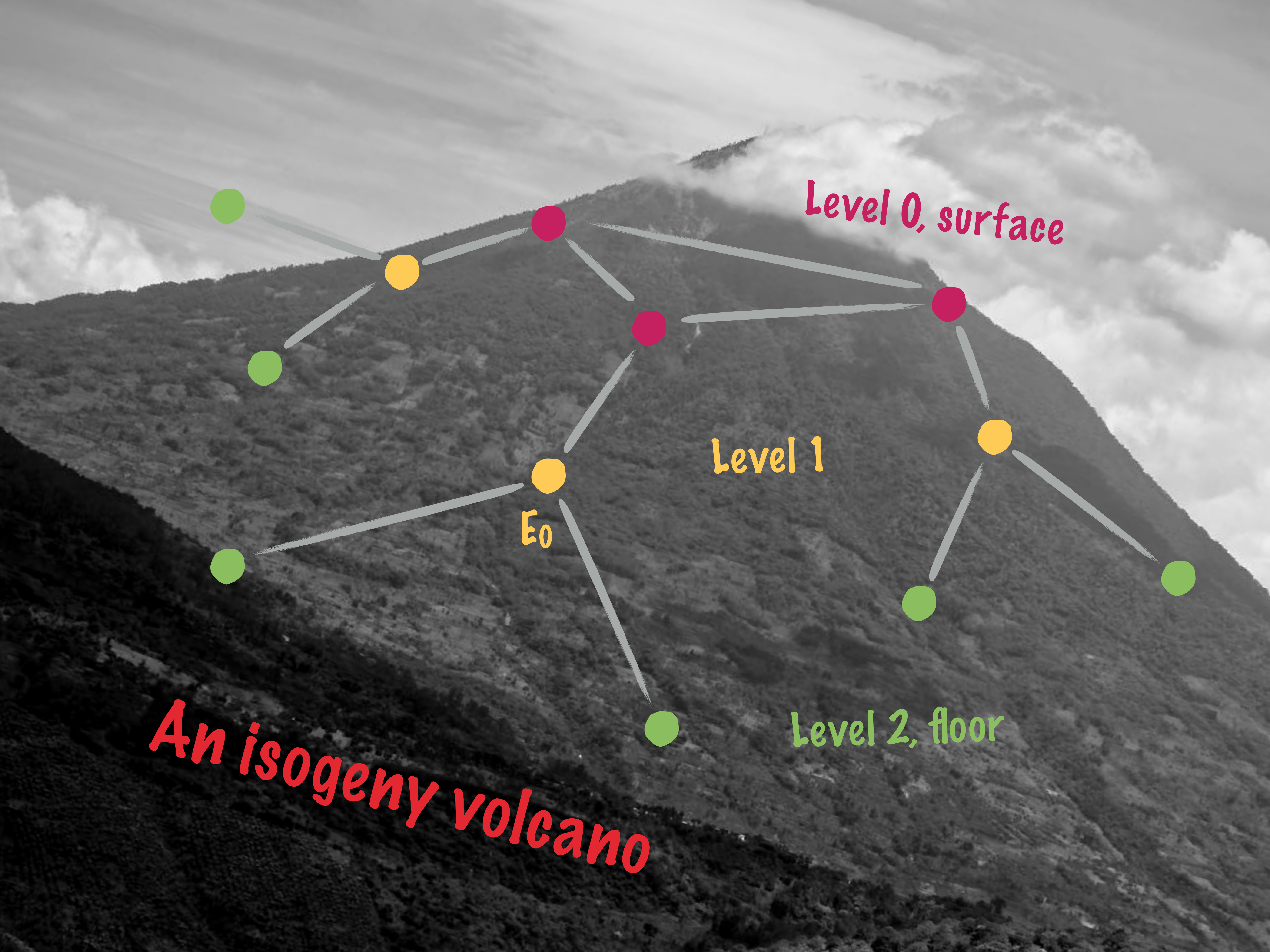


Disjoint isomorphic copies of a tree rooted on the cycle

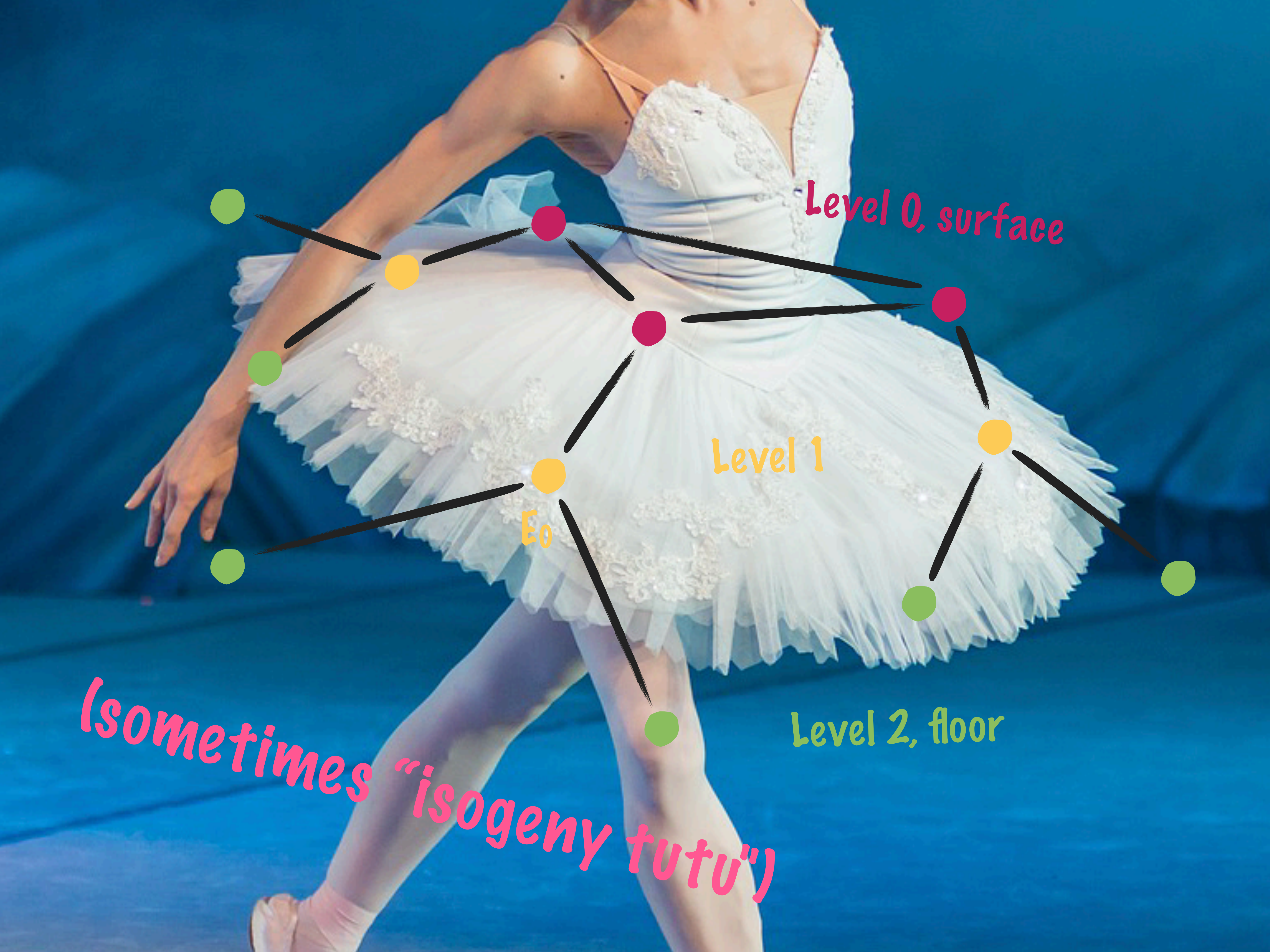
This one is a typical example!

ISOGENY GRAPHS OF ORDINARY ELLIPTIC CURVES





Level 0, surface
Level 1
E_0
Level 2, floor
An isogeny volcano



Level 0, surface
Level 1
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Level 2, floor
(sometimes "isogeny tutu")

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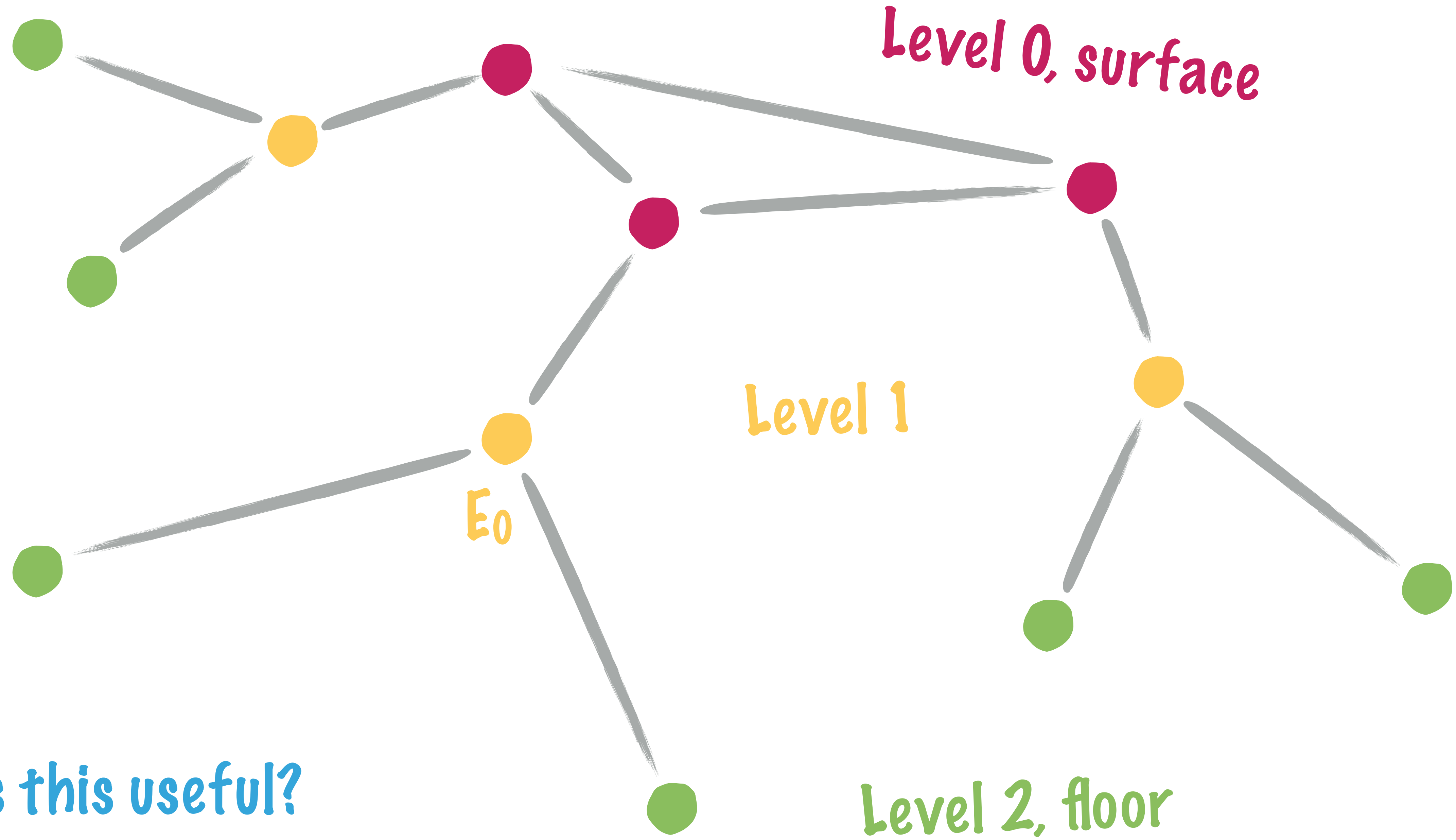

Why is this useful?

By inspecting **solely** the structure of the graph, one can infer that E_0 is at "level 1" in ℓ… which tells a lot about the endomorphism ring of E_0!

APPLICATIONS

- Computing the endomorphism ring of an elliptic curve [Kohel, 1996],
- Counting points [Fouquet et Morain, 2002],
- Random self-reducibility of the discrete logarithm problem [Jao et al., 2005] (worst case to average case reduction)
- Accelerating the CM method [Sutherland 2012],
- Computing modular polynomials [Bröker et al., 2012]

GENERALISING TO ORDINARY ABELIAN VARIETIES…

- These applications motivate the search for a generalisation to other abelian varieties…

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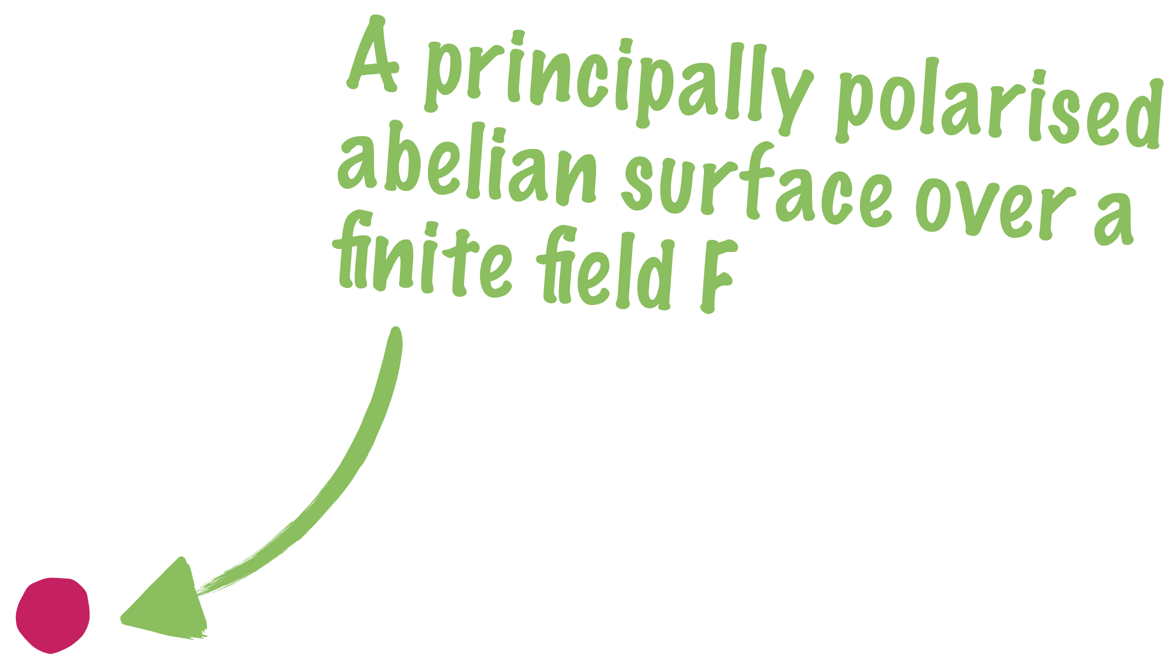

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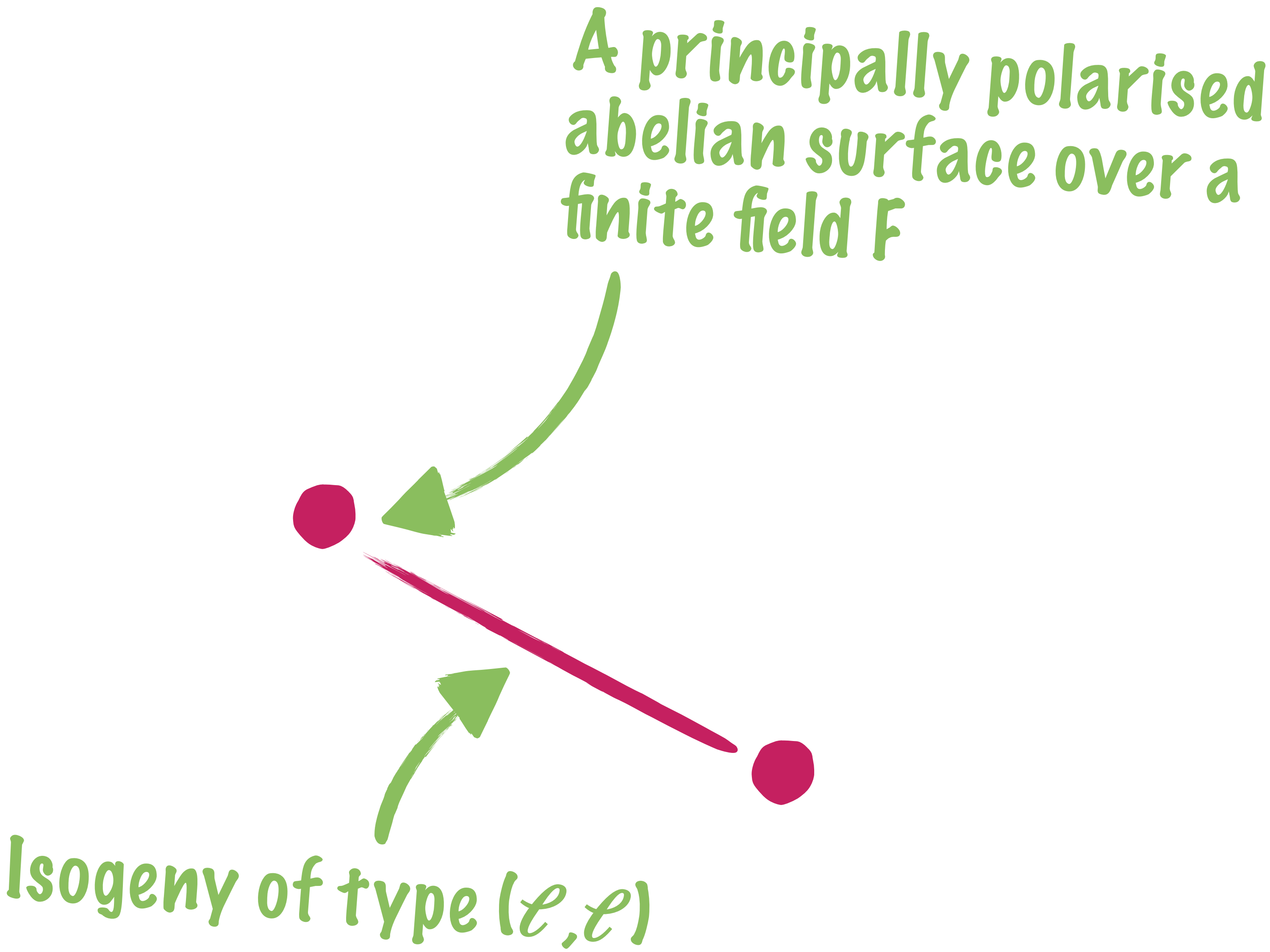

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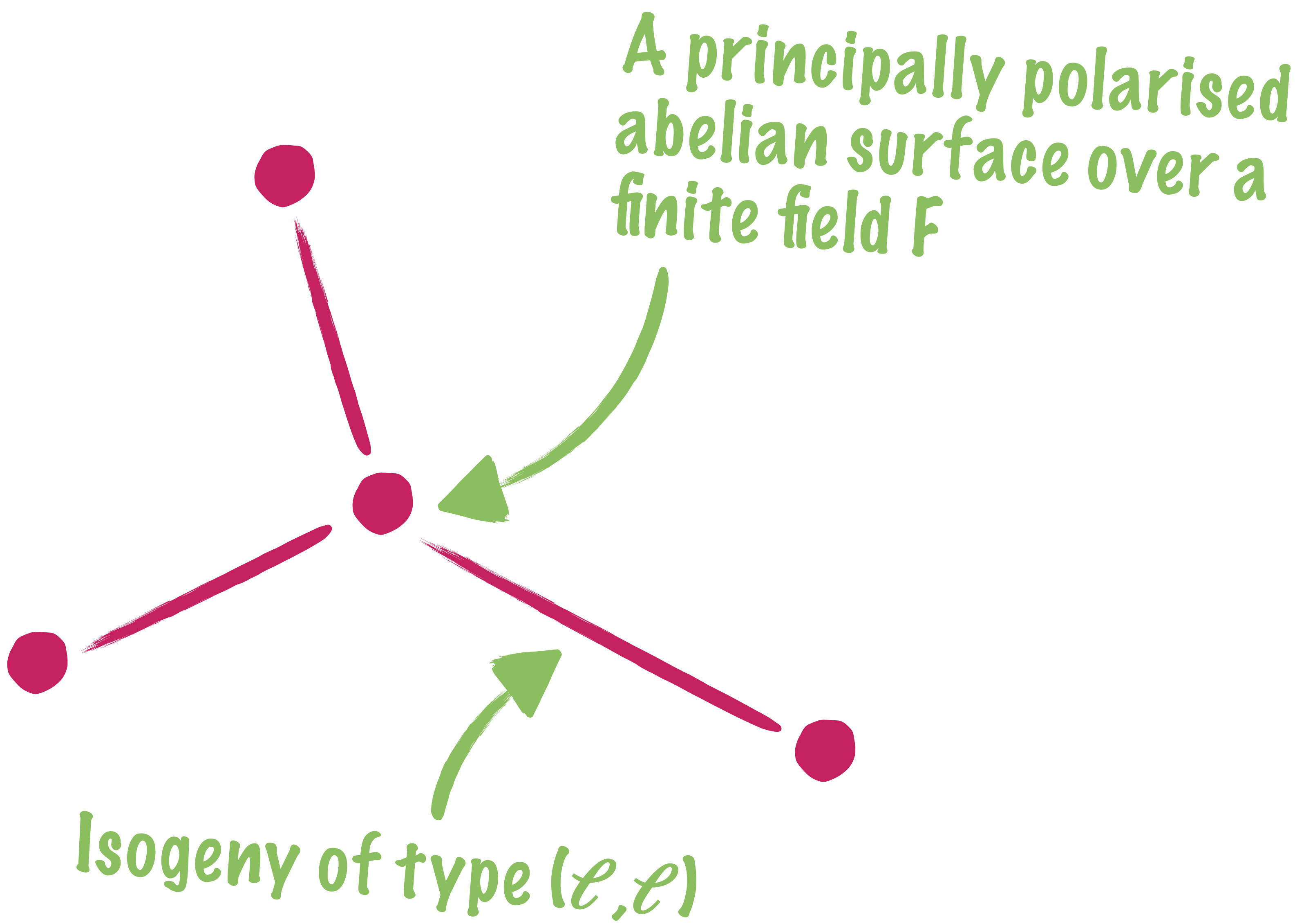

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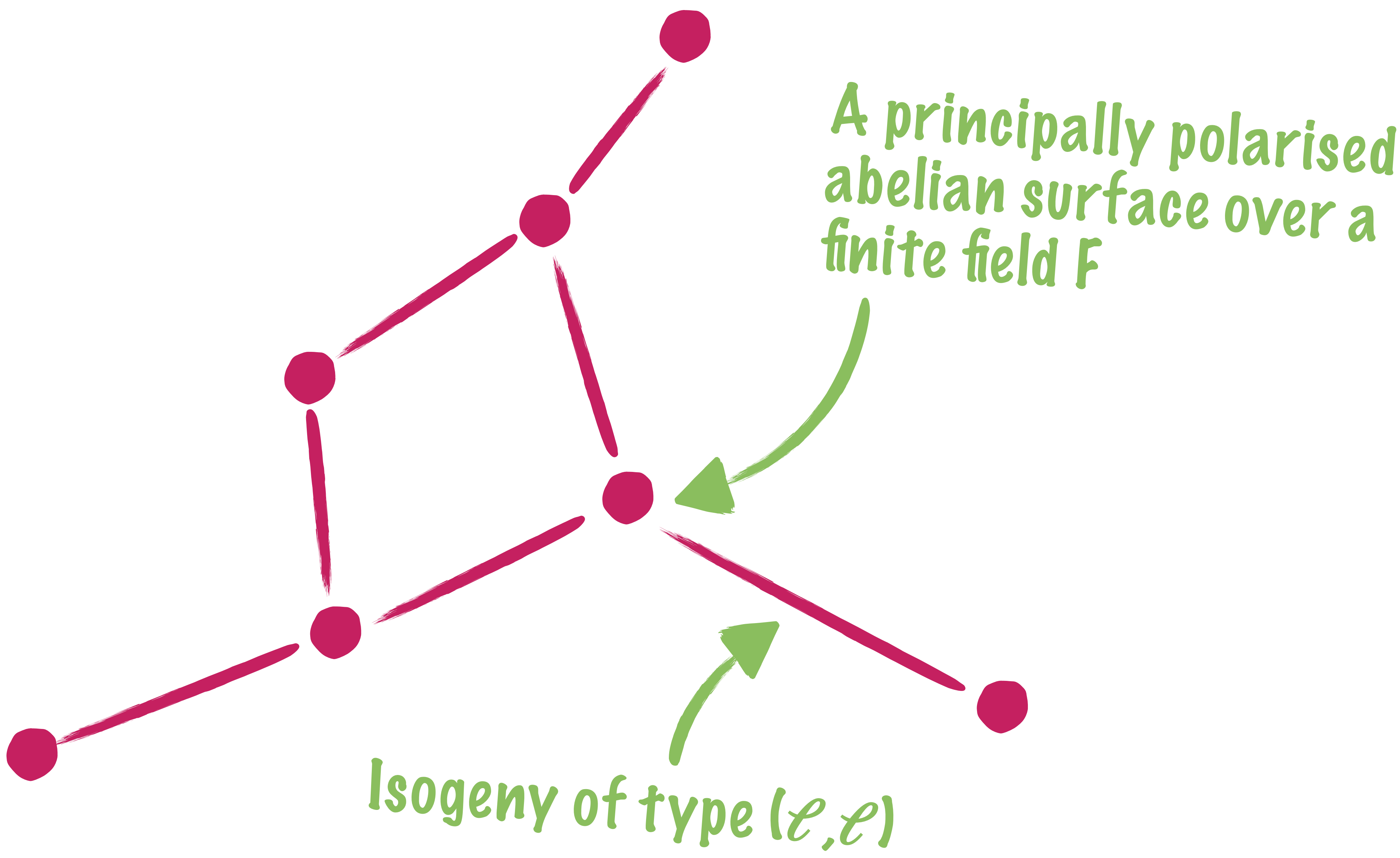

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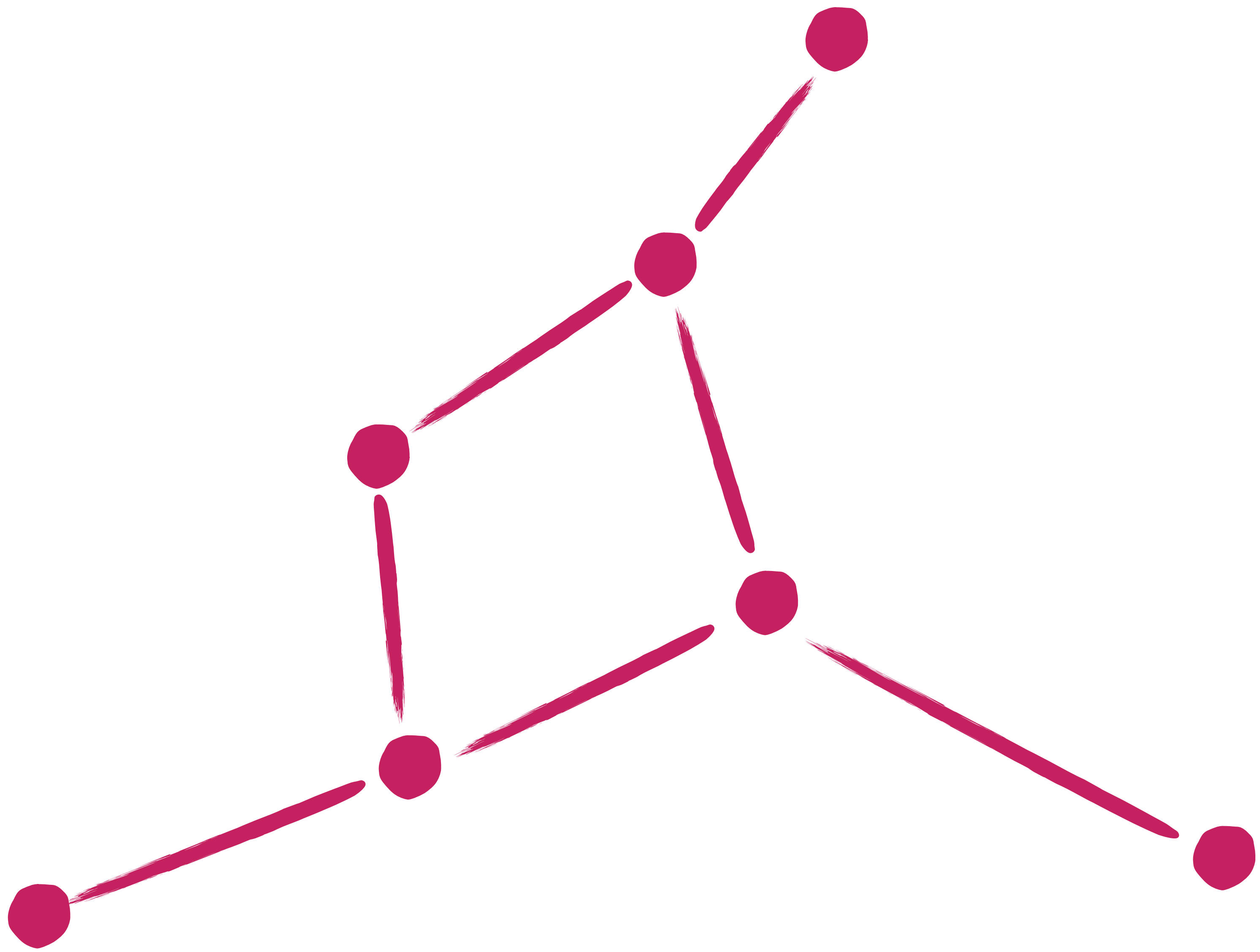

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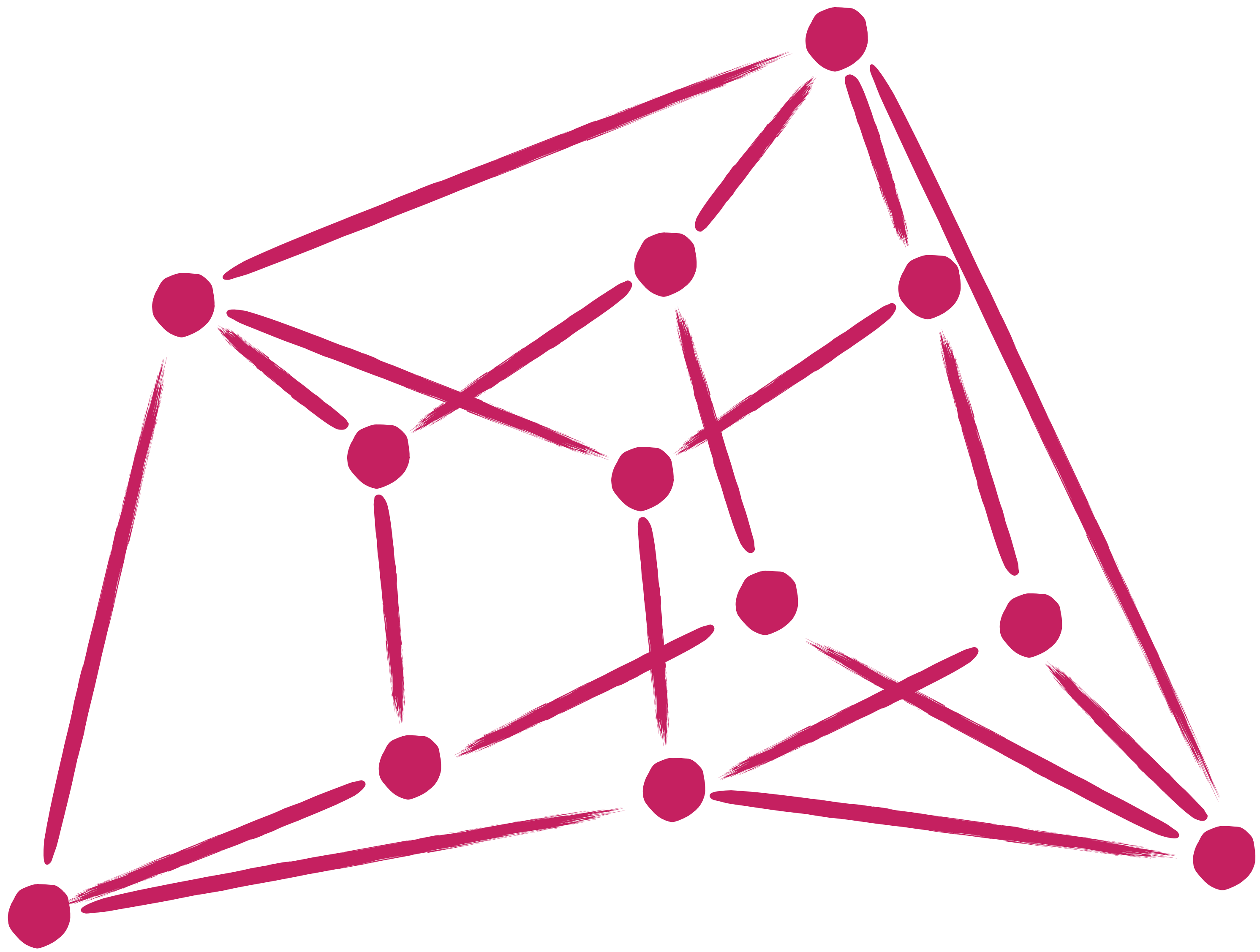

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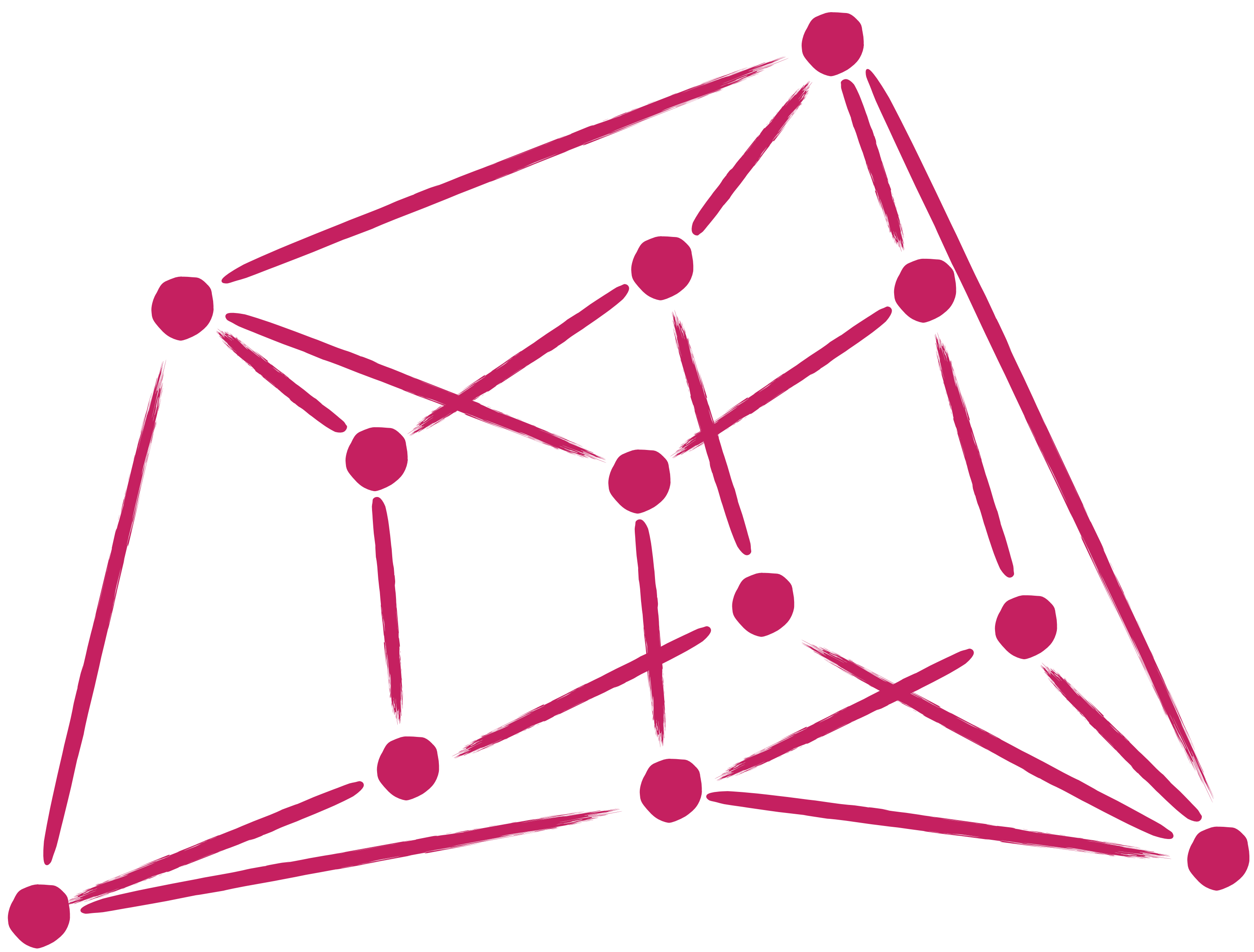

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GENERALISING TO ORDINARY ABELIAN VARIETIES…

Maybe we shouldn't focus on (ℓ,ℓ)-isogenies?

Maybe we do not look for the correct structures?

Should we focus on subgraphs?

ENDOMORPHISM RINGS

ENDOMORPHISM RING AND ALGEBRA

- Let $\mathscr{A}$ be an ordinary abelian variety of dimension g over a finite field $F = \mathbb{F}_q$.

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$$\begin{array}{rl} & K \supset \mathscr{O} \cong \operatorname{End}(\mathscr{A}) \\ 2 & \Big| \\ & K_0 \\ g & \Big| \\ & \mathbb{Q} \end{array}$$

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- The algebra $K = \mathrm{End}(\mathscr{A}) \otimes \mathbb{Q}$ is a number field of degree $2g$ (a CM-field).
- $\mathrm{End}(\mathscr{A})$ is isomorphic to an order $\mathscr{O}$ of K (i.e., a lattice of dimension $2g$ in K, that is also a subring).

$$\begin{tikzcd}
K \supset \mathscr{O} \cong \mathrm{End}(\mathscr{A}) \arrow[d, dash, "2"'] \\
K_0 \arrow[d, dash, "g"'] \\
\mathbb{Q}
\end{tikzcd}$$

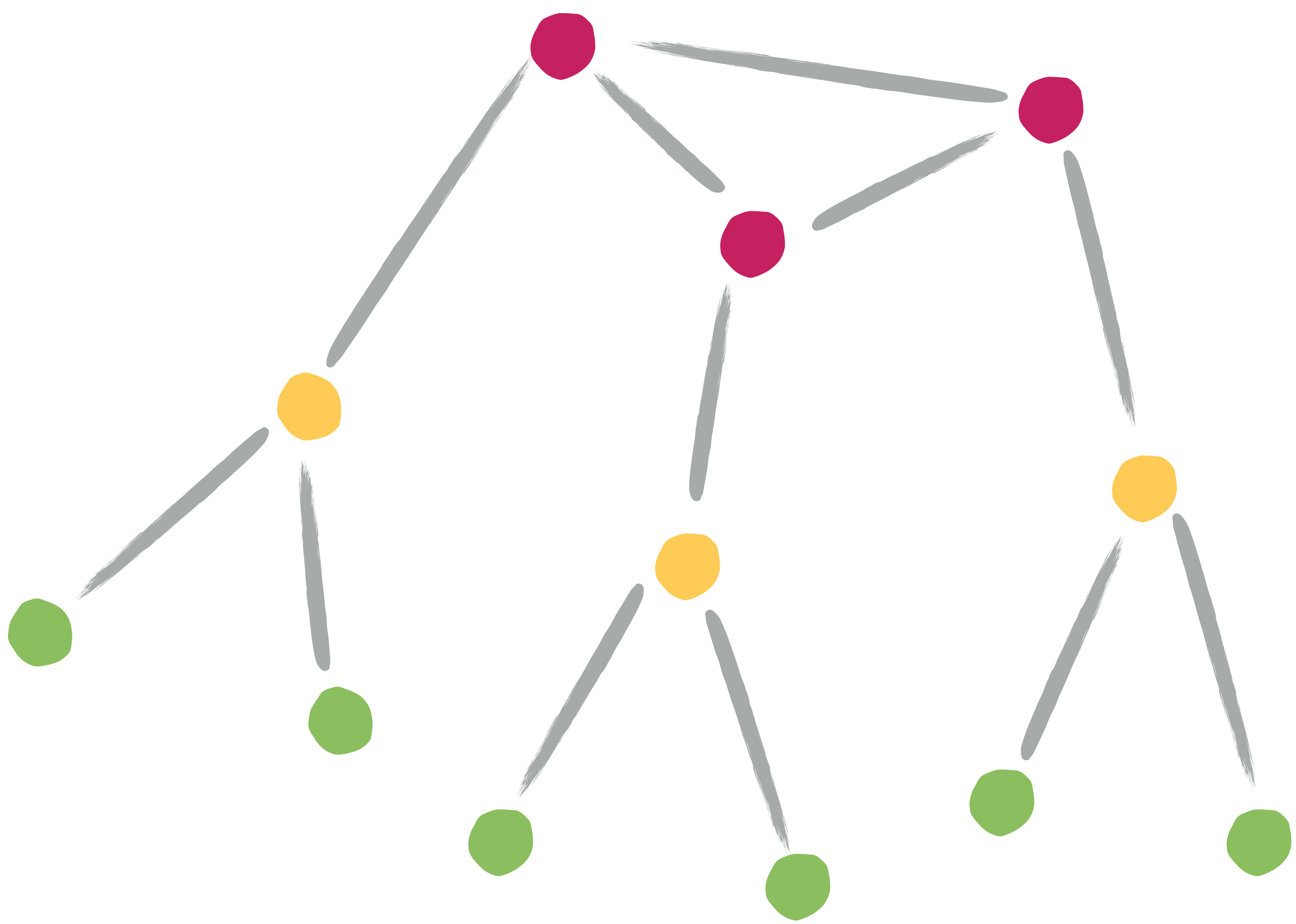
THE CASE OF ELLIPTIC CURVES

- If $\mathscr{A} = E$ is an elliptic curve, the dimension is $g = 1$.
- K has a **maximal order** $\mathcal{O}_K$, the ring of integers of K.
- Any order of K is of the form
$$\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K,$$
for a positive integer f, the **conductor**.

$$\begin{array}{l} K \supset \mathcal{O} \cong \mathrm{End}(E) \\ \quad \Big| \, 2 \\ K_0 = \mathbb{Q} \end{array}$$

THE CASE OF ELLIPTIC CURVES

The “levels” of the volcano of ℓ-isogenies tell how many times ℓ divises the conductor. Here, $(f,\ell) = 1$.



End $\cong \mathbb{Z} + f\mathcal{O}_K$

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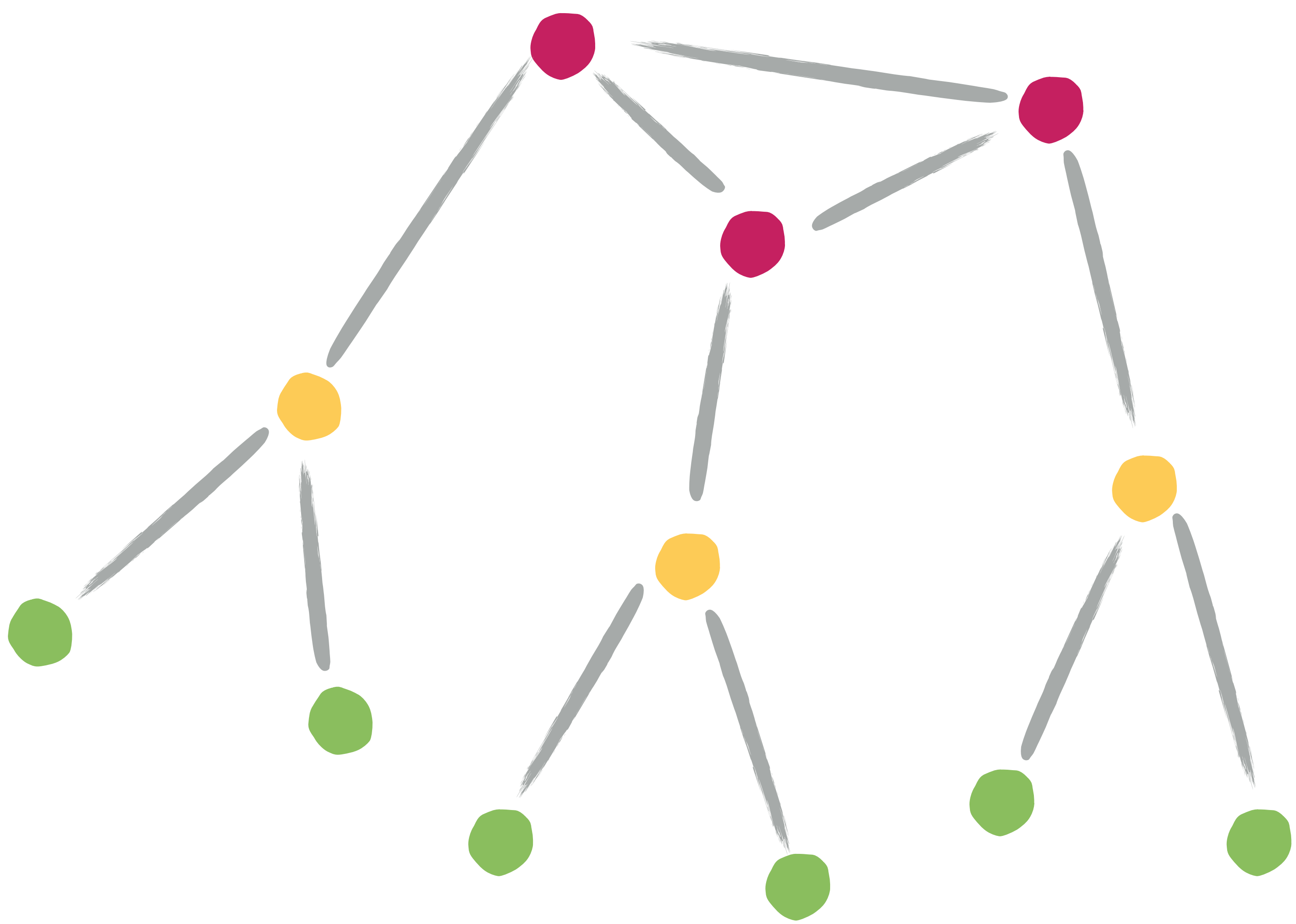
THE CASE OF ELLIPTIC CURVES

Only an ℓ-isogeny can change the valuation at ℓ of the conductor.



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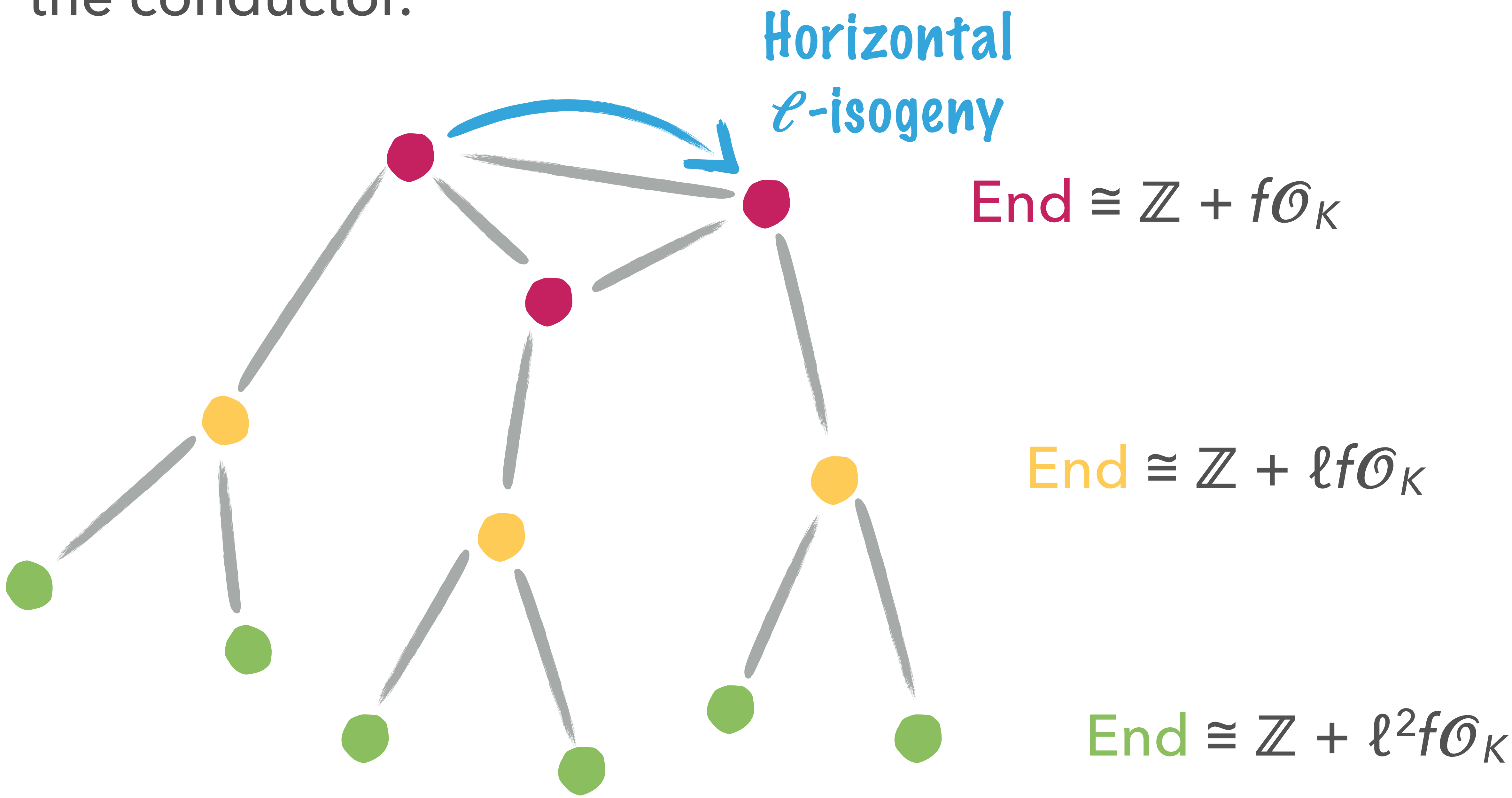

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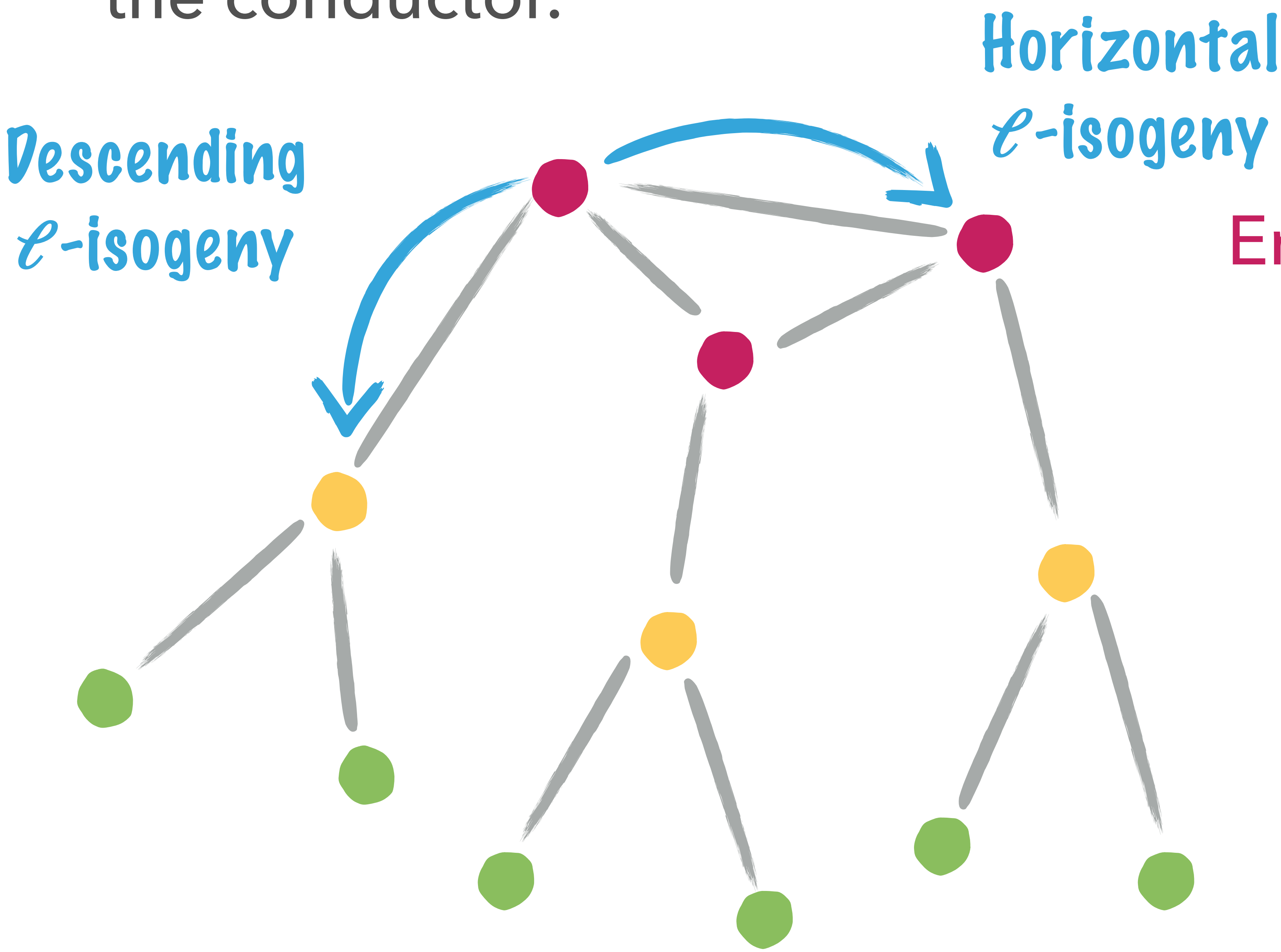

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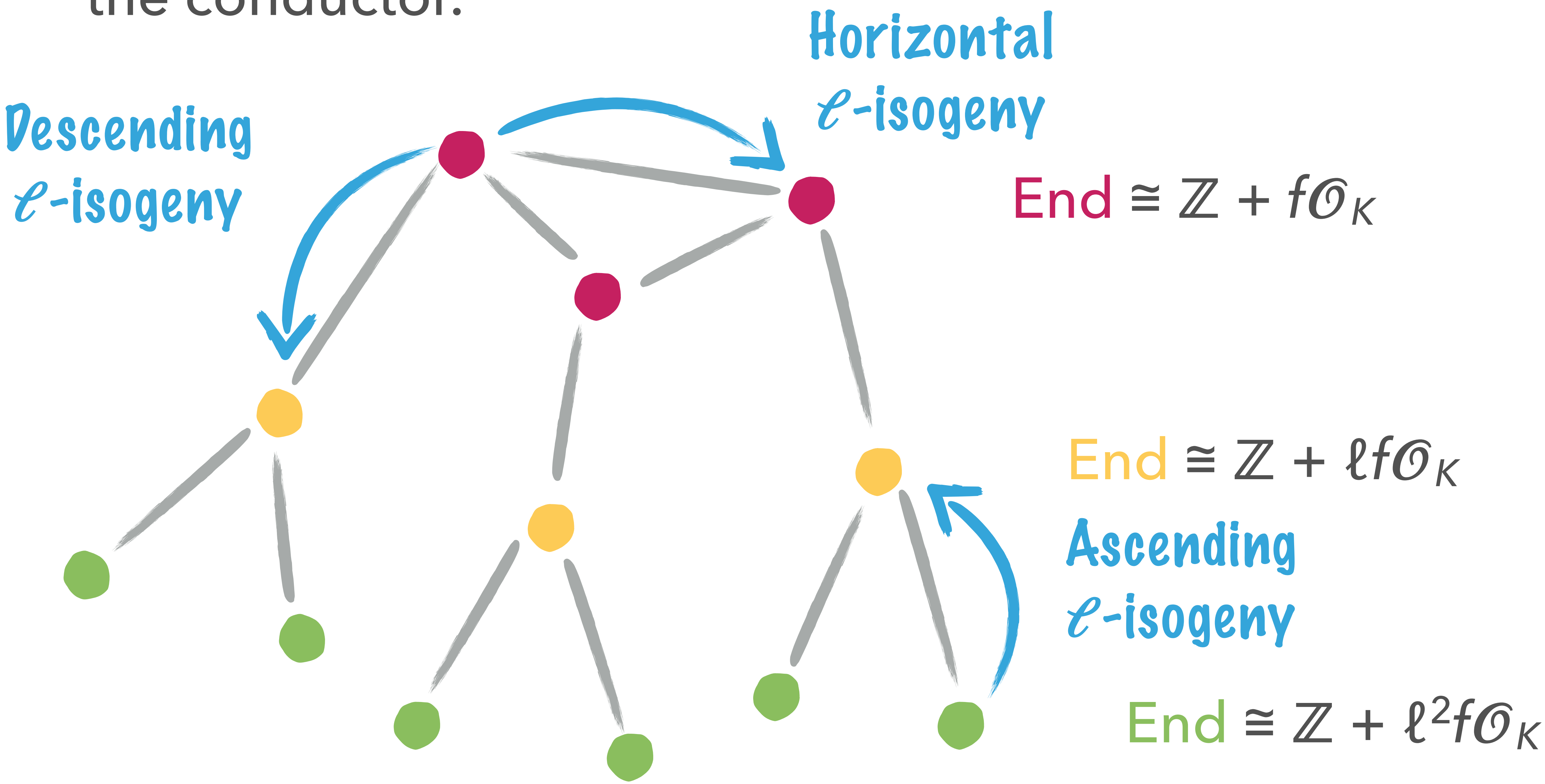

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We actually look at this result "localy" at a prime ℓ, i.e., for the étale algebra $K \otimes \mathbb{Q}_\ell$.

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- For $K_0 = \mathbb{Q}$, any order has maximal RM since $\mathcal{O}_{K_0} = \mathbb{Z}$.

VOLCANOES AGAIN

ℓ-ISOGENIES

- For an elliptic curve, the conductor is an integer f, which decomposes as a product of prime numbers: we then look at ℓ-isogenies where ℓ is a prime number

$\mathfrak{l}$-ISOGENIES

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- For $g > 1$ and maximal RM, the conductor is an ideal $\mathfrak{f}$ of $\mathcal{O}_{K_0}$, and decomposes into prime ideals…

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An $\mathfrak{l}$-isogeny from $\mathscr{A}$ is an isogeny whose kernel is a cyclic sub-$\mathcal{O}_{K_0}$-module of $\mathscr{A}[\mathfrak{l}]$.

Only an $\mathfrak{l}$-isogeny can change the valuation at $\mathfrak{l}$ of the conductor.

VOLCANOES AGAIN?

If $\mathscr{A}$ has maximal RM (locally at ℓ), and $\mathfrak{l}$ is a prime ideal of $\mathscr{O}_{K_0}$ above ℓ, is the graph of $\mathfrak{l}$-isogenies a volcano?

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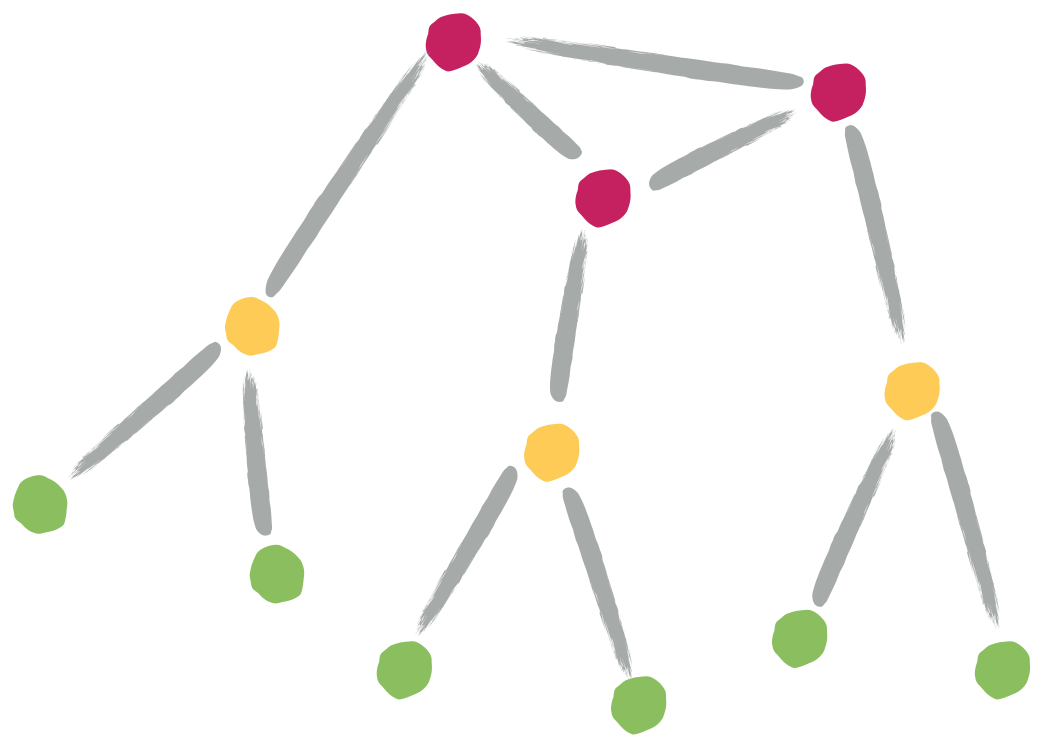
Theorem: yes!... at least when $\mathfrak{l}$ is principal, and all the units of $\mathcal{O}_K$ are totally real!



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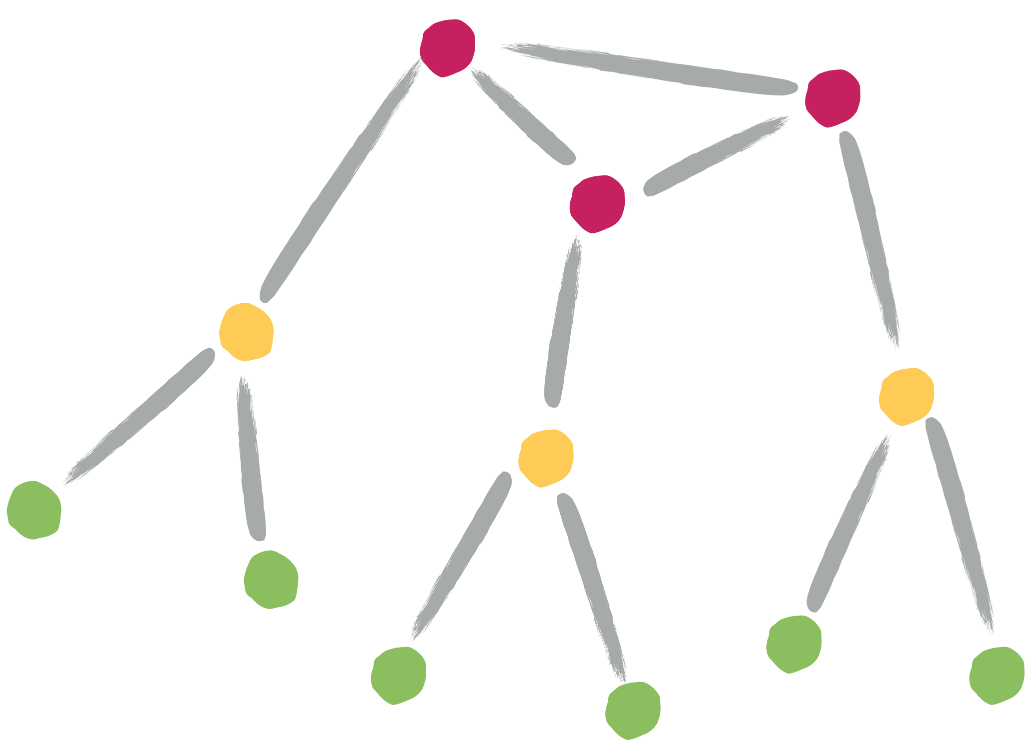

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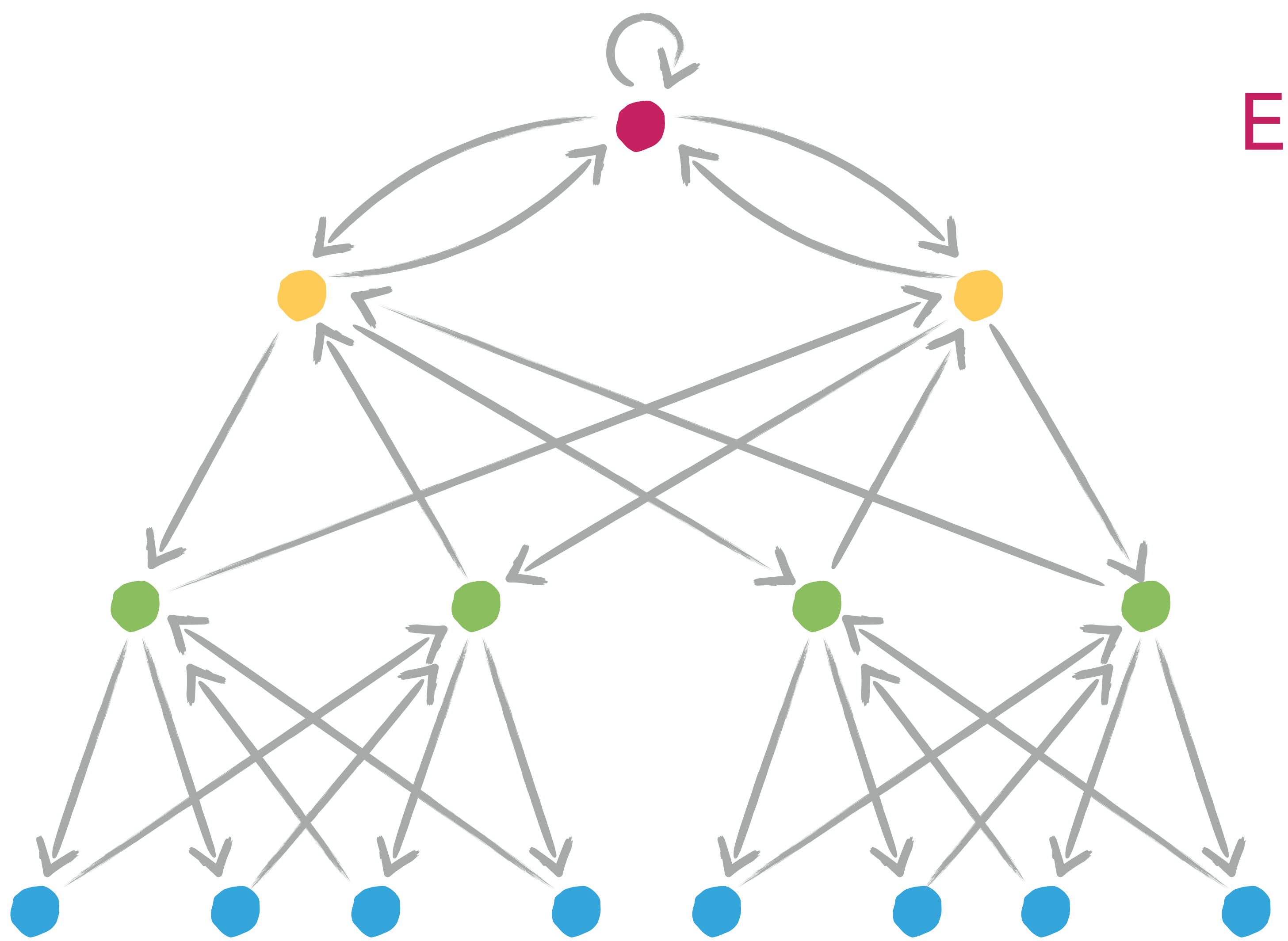
VOLCANOES AGAIN?

If $\mathfrak{l}$ is not principal? The graph is oriented!



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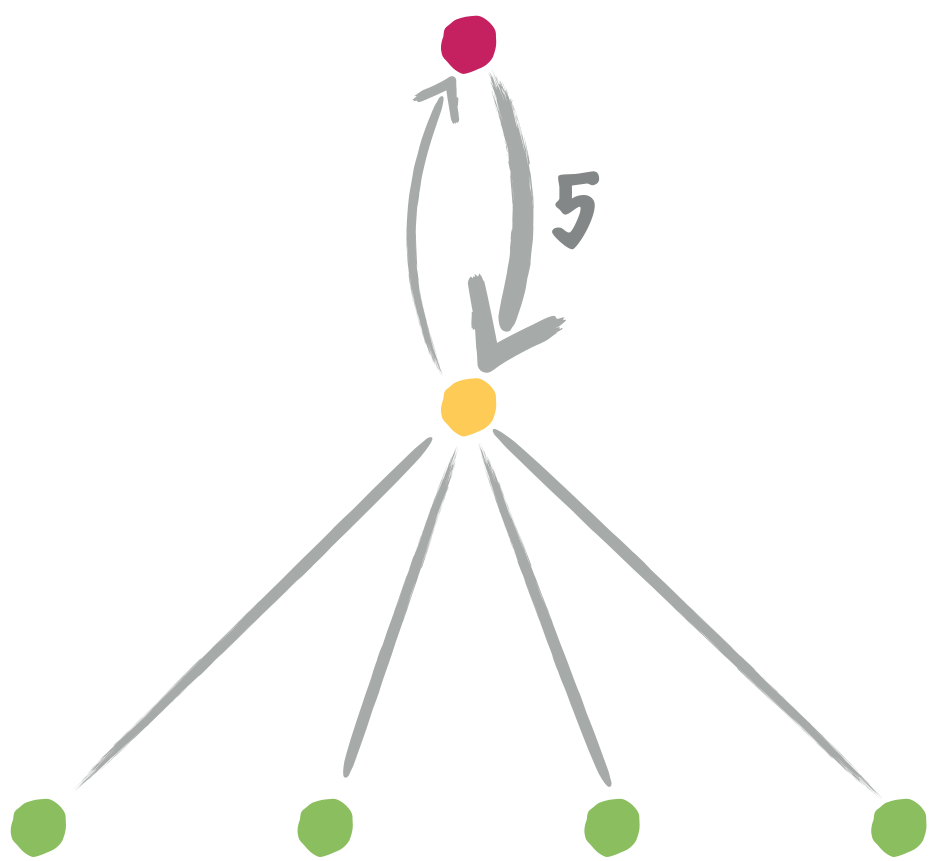
VOLCANOES AGAIN?

If $\mathcal{O}_K$ has complex units? Multiplicities appear

For instance, $K = \mathbb{Q}(\zeta_5)$, $K_0 = \mathbb{Q}\left(\zeta_5 + \zeta_5^{-1}\right)$, and $\mathfrak{l} = 2\mathcal{O}_{K_0}$.



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IN DIMENSION 2

(ℓ,ℓ)-ISOGENIES

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- Let $\mathscr{A}$ be a principally polarised, ordinary abelian surface.
- An (ℓ,ℓ)-isogeny is an isogeny $\mathscr{A} \to \mathscr{B}$ whose kernel is a maximal isotropic subgroup of $\mathscr{A}[\ell]$ for the Weil pairing.
- (ℓ,ℓ)-isogenies are easier to compute! Much more efficient than $\mathfrak{l}$-isogenies…

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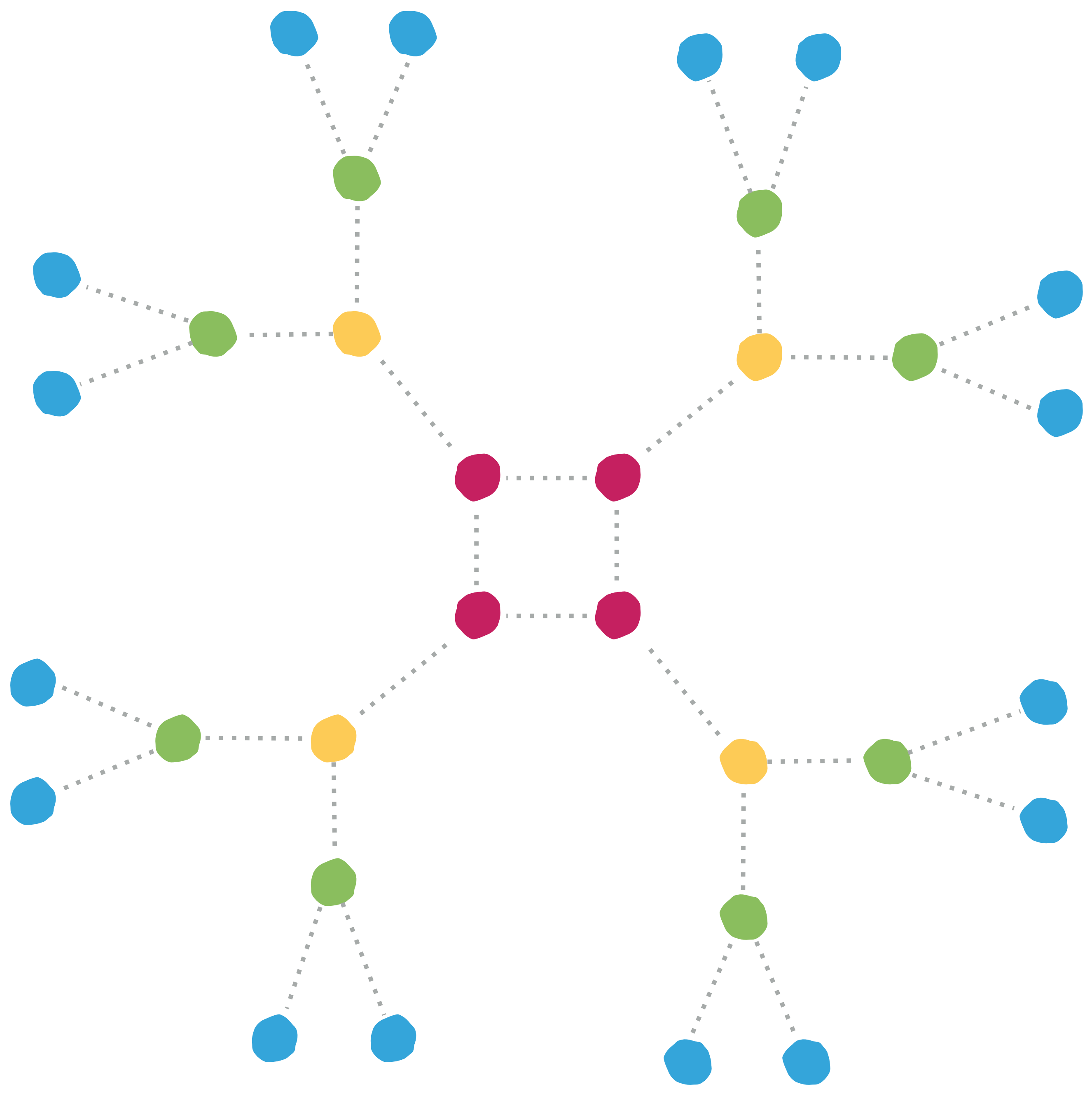
We show that (ℓ,ℓ)-isogenies preserving the maximal RM are exactly:

- The $\mathfrak{l}$-isogenies if ℓ is inert in K_0 (i.e., $\mathfrak{l} = \ell\mathcal{O}_{K_0}$)
- The compositions of an $\mathfrak{l}_1$-isogeny with an $\mathfrak{l}_2$-isogeny if ℓ splits or ramifies as $\ell\mathcal{O}_{K_0} = \mathfrak{l}_1\mathfrak{l}_2$.

GRAPHS OF (ℓ,ℓ)-ISOGENIES PRESERVING THE RM



Assume $\ell\mathcal{O}_{K_0} = \mathfrak{l}^2$

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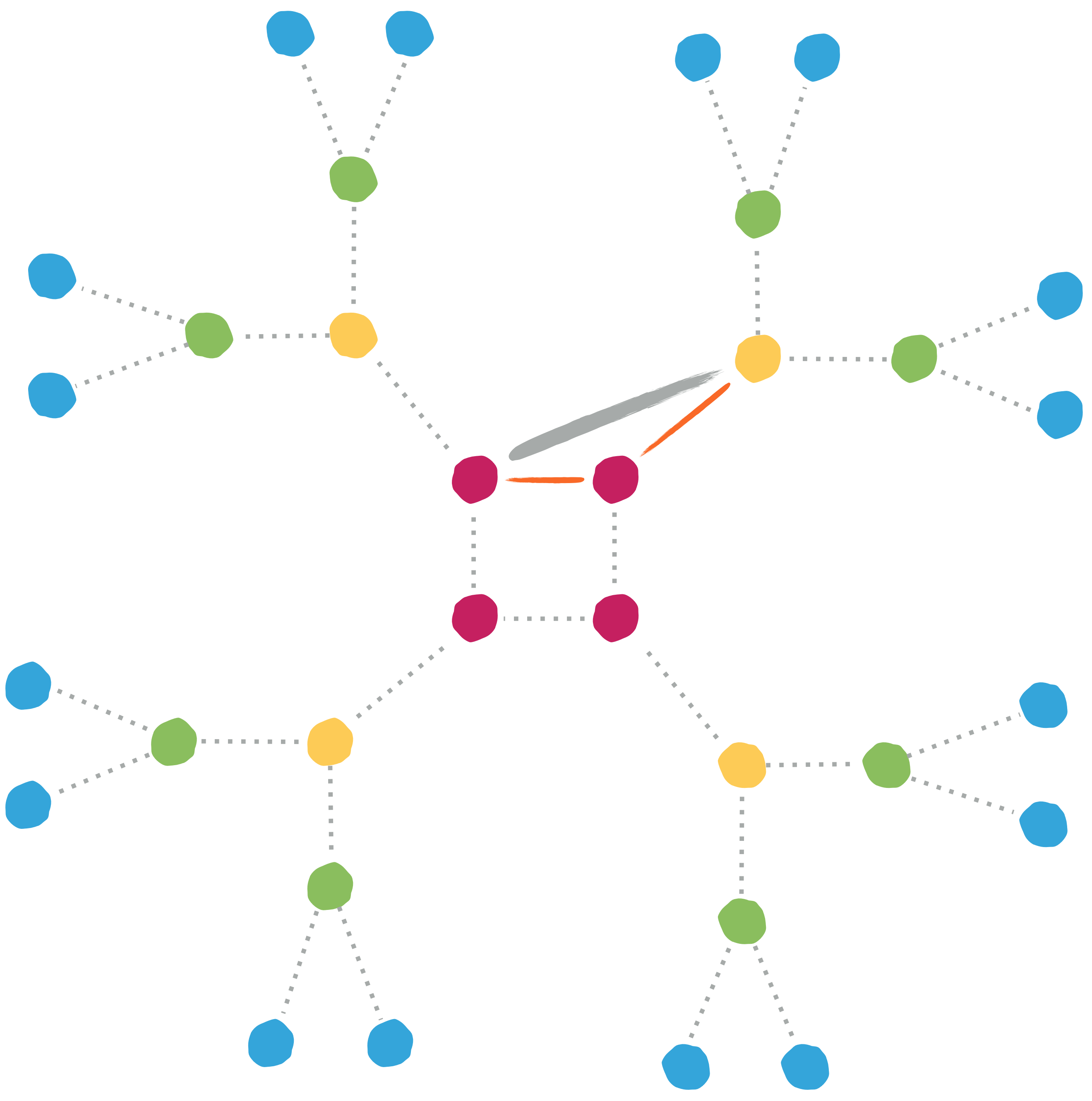

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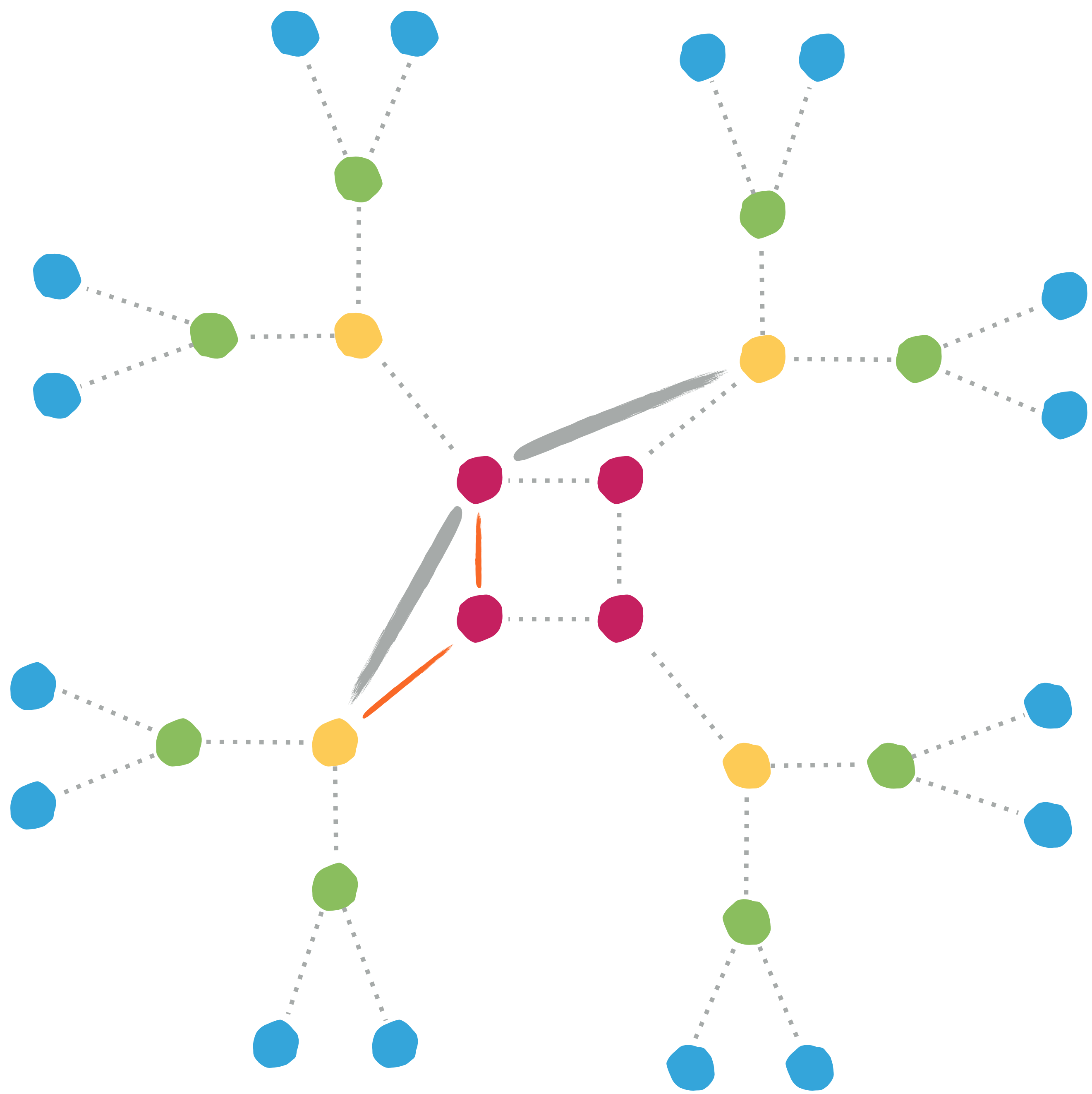

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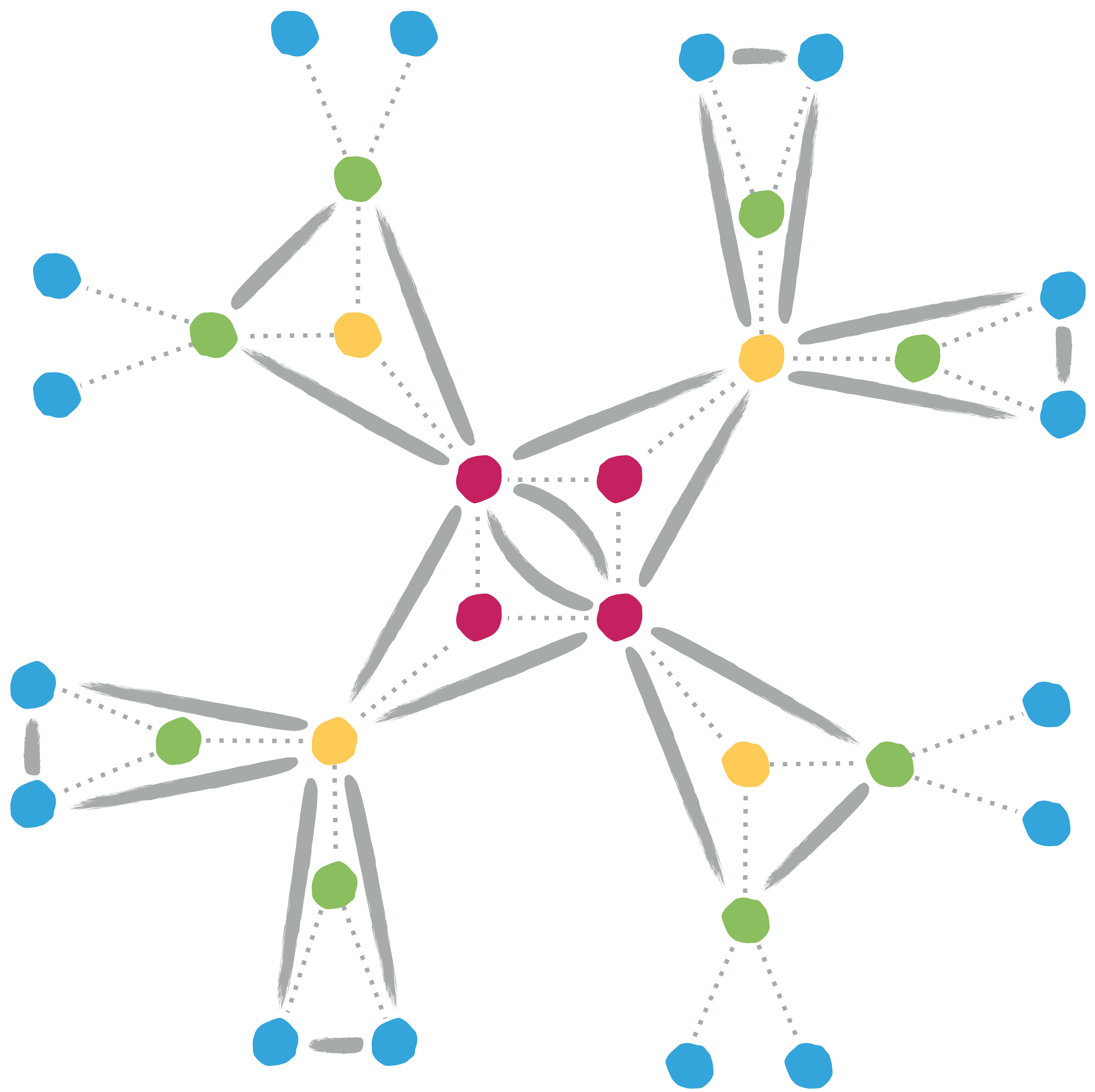

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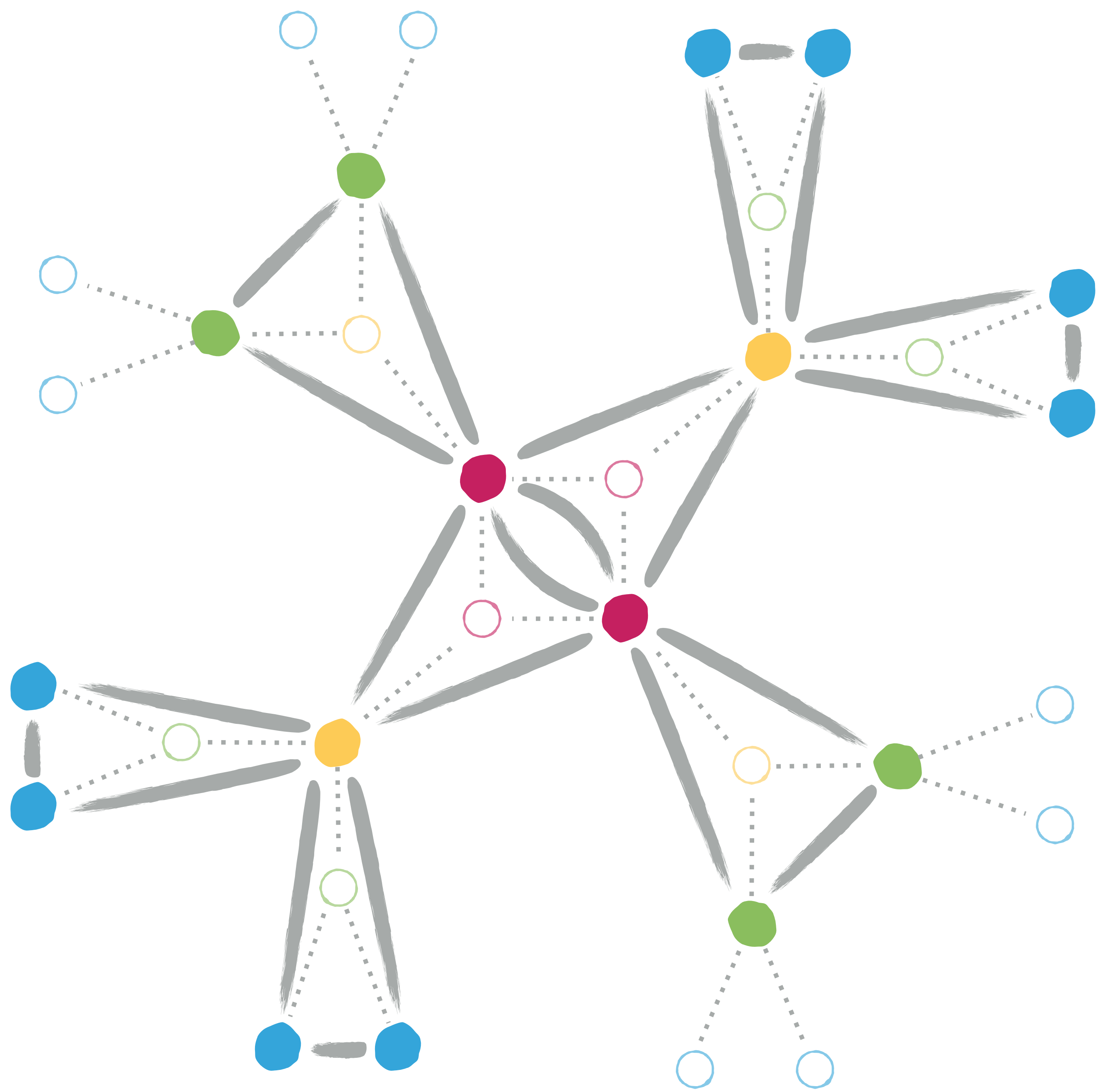

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WHERE TO GO FROM THERE?

- We described the structure of graphs of (ℓ,ℓ)-isogenies preserving the maximal RM.
- It is also interesting to look at (ℓ,ℓ)-isogenies changing the RM. We can describe this graph locally.
- In particular, if the RM is not maximal, we show that there is an (ℓ,ℓ)-isogeny increasing it.
- A first application: these results allow to describe an algorithm finding a path of (ℓ,ℓ)-isogenies to a variety with maximal endomorphism ring.

TECHNIQUES

ℓ-ADIC LATTICES AND COMPLEX MULTIPLICATION

THE TATE MODULE

- We have the following sequence of morphisms

$$0 \xleftarrow{\ell} \mathscr{A}[\ell] \xleftarrow{\ell} \mathscr{A}[\ell^2] \xleftarrow{\ell} \mathscr{A}[\ell^3] \xleftarrow{\ell} \mathscr{A}[\ell^4] \xleftarrow{\ell} \ldots$$

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- Given a lattice L in V containing T, the set of elements of K_ℓ preserving L is an order in K_ℓ, denoted $\mathcal{O}(L)$.

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- We can study isogenies and their relation to endomorphism rings by looking at lattices in the ℓ-adic vector space V.

ℓ-ADIC LATTICES AND $\mathfrak{l}$-ISOGENIES

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- $\mathcal{O}^\times = (\mathrm{End}(\mathscr{A}) \otimes \mathbb{Z}_\ell)^\times$ acts on $\mathbb{P}^1(\mathcal{O}/\mathfrak{l}\mathcal{O})$, and elements that are **not** fixed by this action are descending $\mathfrak{l}$-isogenies.

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the surface

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Is this isogeny ascending?

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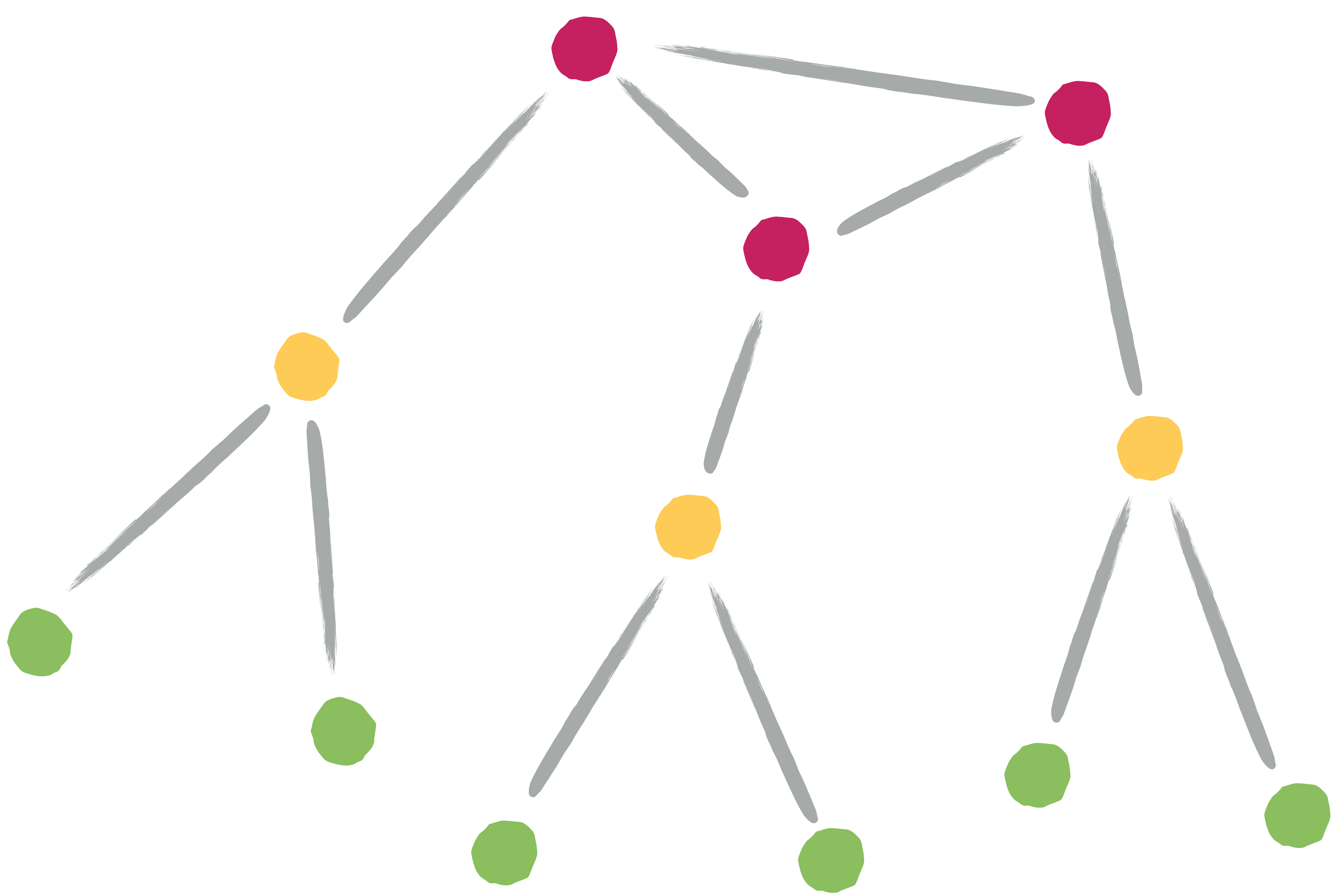

$\mathrm{End} \cong \mathcal{O}_{K_0} + \mathfrak{f}\mathcal{O}_K$

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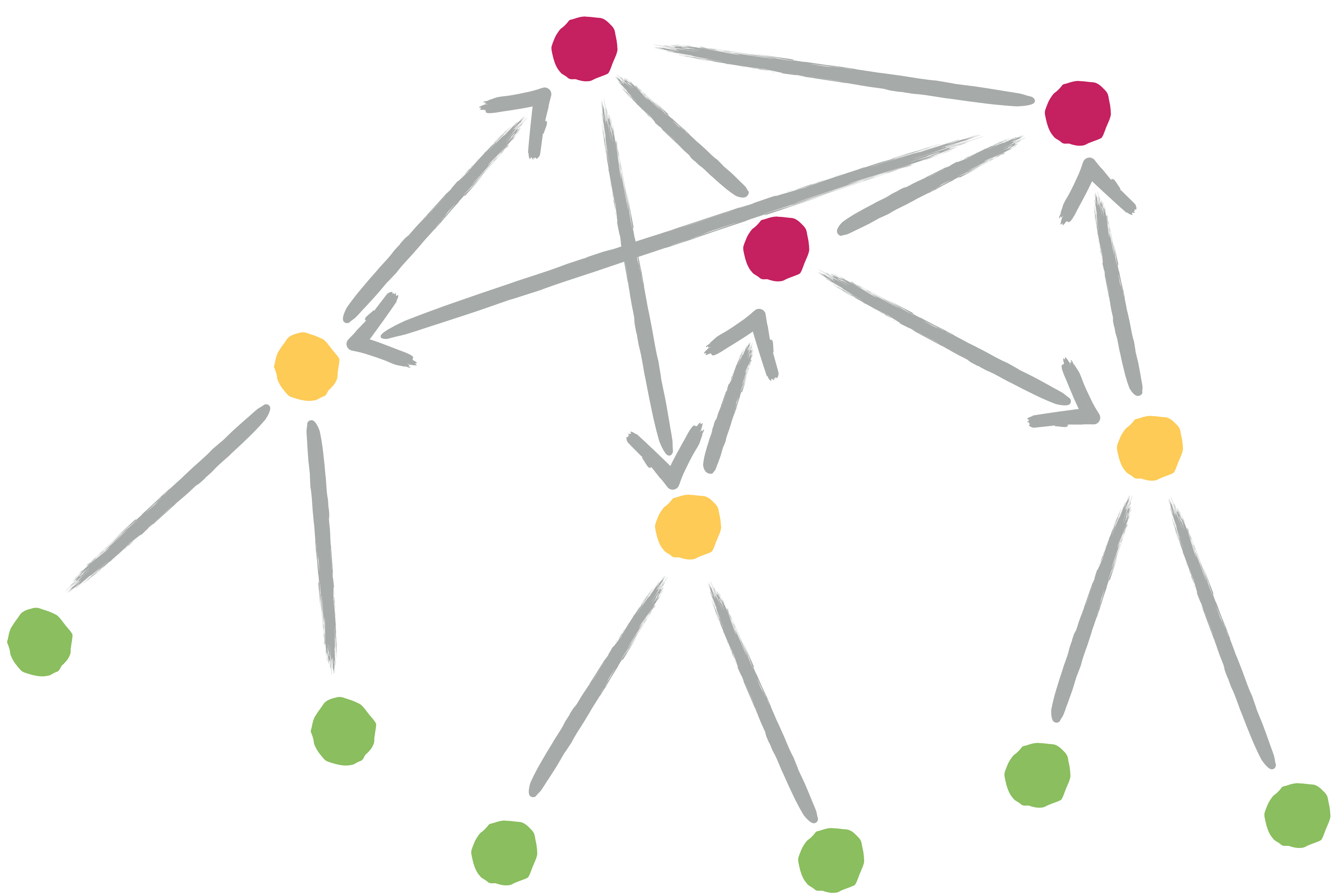

Why not this?

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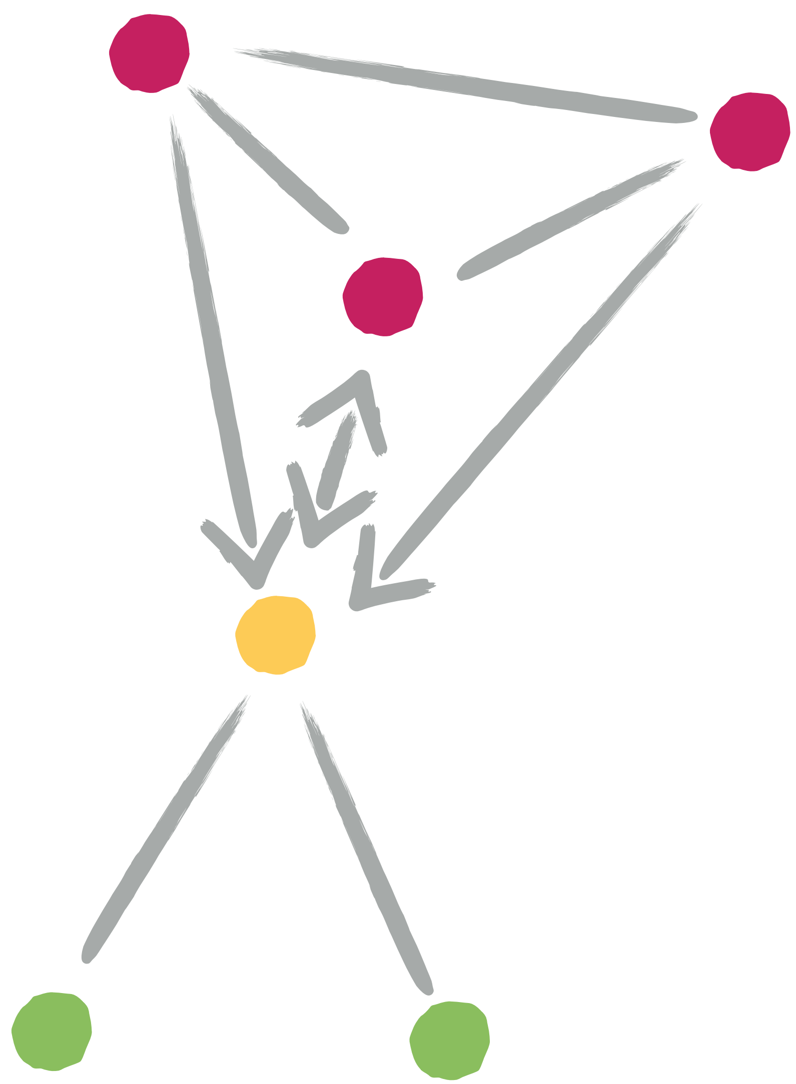

Or this?

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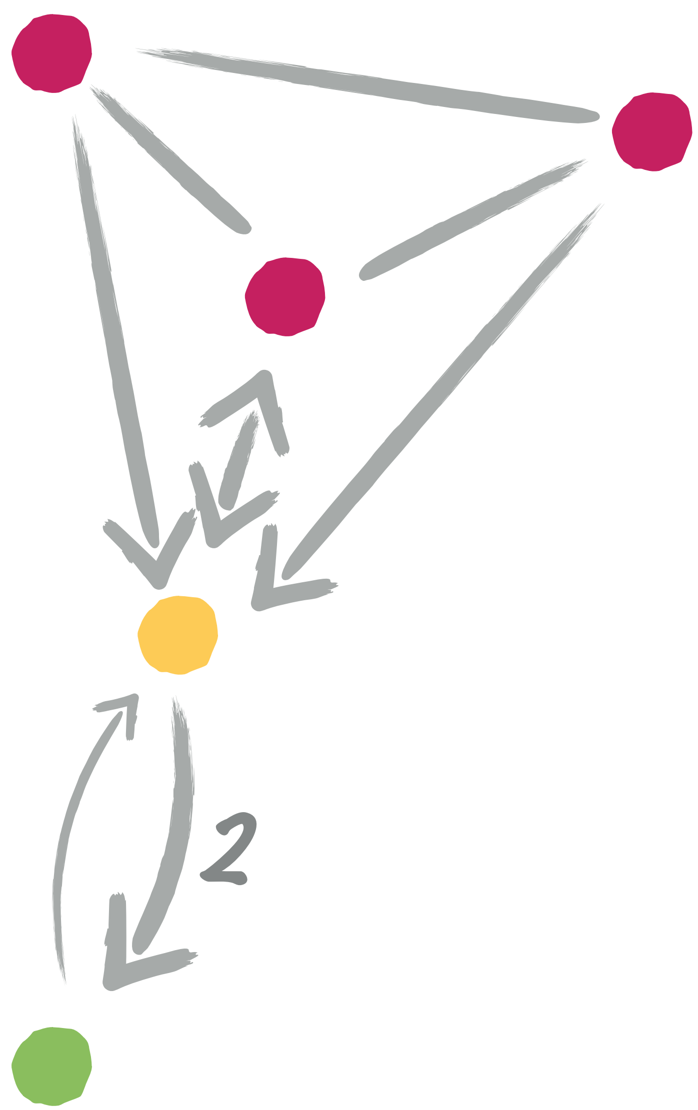

What about this?

DESCENDING, THEN ASCENDING

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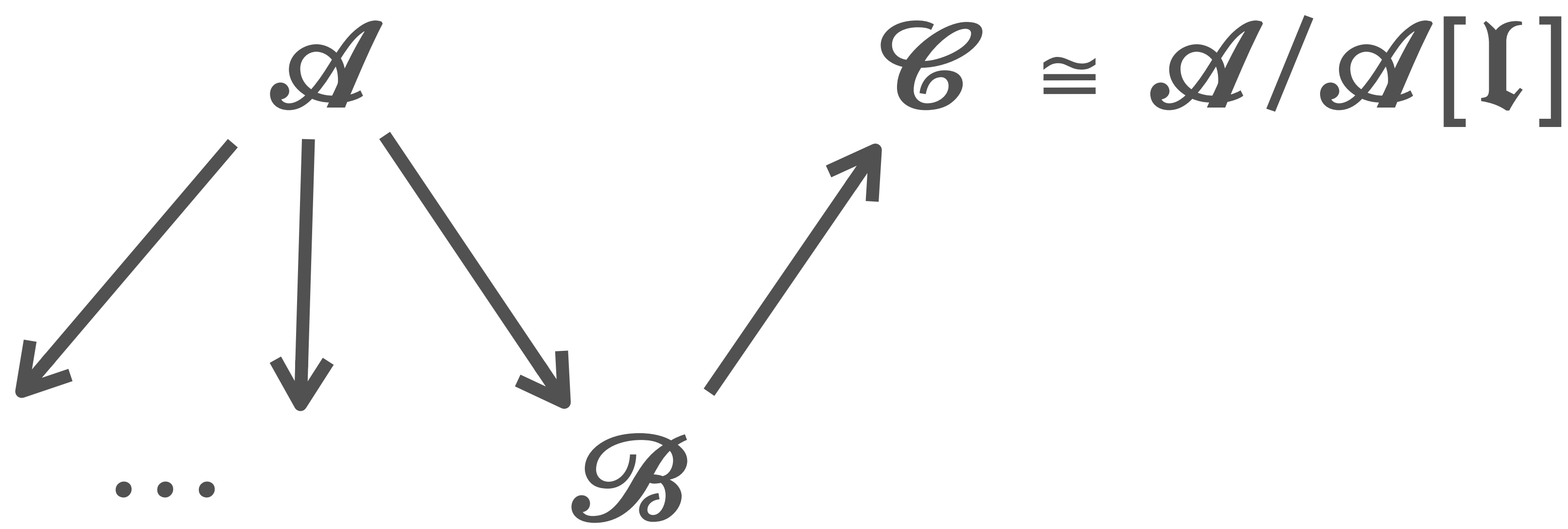

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- The index $[\mathrm{End}(\mathscr{A})^\times : \mathrm{End}(\mathscr{B})^\times]$ is always 1 if all the units of K are totally real (it is the case of any quartic $K \neq \mathbb{Q}(\zeta_5)$)

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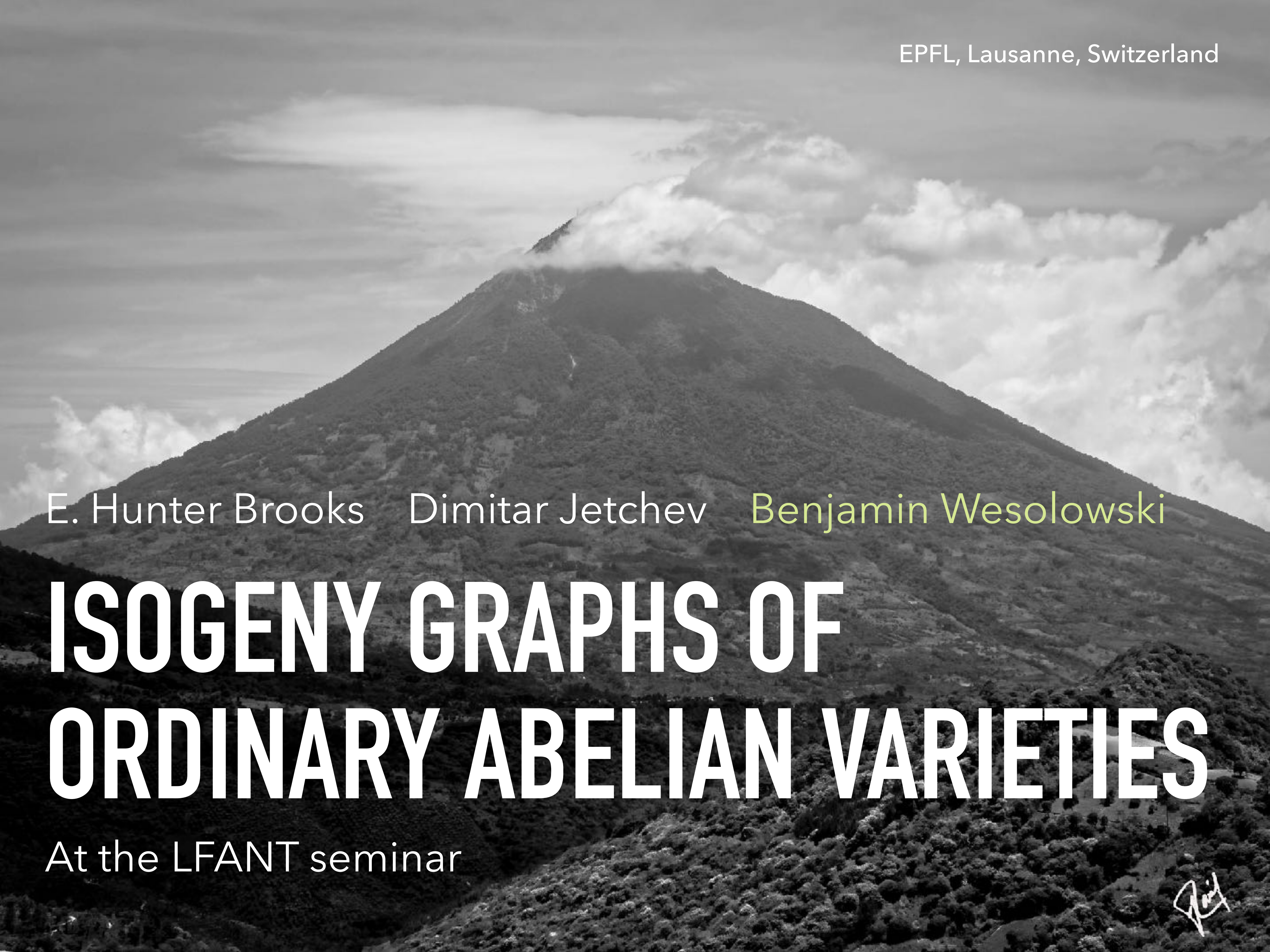
- Putting all this together, we obtain a precise description of the isogeny graphs.
- They are volcanoes exactly when K has no complex units (no multiplicities on the edges) and $\mathfrak{l}$ is principal (the edges are undirected).



EPFL, Lausanne, Switzerland

E. Hunter Brooks Dimitar Jetchev Benjamin Wesolowski

ISOGENY GRAPHS OF ORDINARY ABELIAN VARIETIES

At the LFANT seminar