Modular Symbols and $p$-adic $L$-functions

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after Pollack and Stevens
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Modular curves

Let $G \subset \text{PSL}_2(\mathbb{Z})$ be a subgroup of finite index. It acts on Poincaré’s upper half plane

$\mathfrak{h} := \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$

by fractional linear transformations:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d},$

as well as on its boundary, the real projective line $\mathbb{P}^1(\mathbb{R})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (p : q) := \frac{ap + bq}{cp + dq}.$

N.B. We identify the point at infinity $(1 : 0)$ with $i\infty$ on the Riemann sphere.

Let $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ be the completed upper half plane. The quotient space $G \backslash \mathfrak{h}$ is compactified by adding a finite number of \textit{cusps} from $G \backslash \mathbb{P}^1(\mathbb{Q})$. The result is the \textit{modular curve} $X(G) = G \backslash \mathfrak{h}^*$, a compact Riemann surface.

Motivating example:

$G = \Gamma_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \};$

in this case, $X(G)$ is the classical modular curve $X_0(N).$
Modular forms (1/2)

For a given integer $k$, the group $G$ act in weight $k$ on functions on $\mathfrak{h}$

$$f |_k \gamma := (cz + d)^{-k} f(\gamma \cdot z), \quad \gamma = \left(\begin{array}{cc}a & b \\ c & d \end{array} \right) \in G.$$  

A modular function of weight $k$ for $G$ is a meromorphic function on $\mathfrak{h}^*$ (on $\mathfrak{h}$ and at all cusps) satisfying

$$f |_k \gamma = f, \quad \forall \gamma \in G.$$  

For instance, a modular function of weight 0 is a function on $X(G)$; a form of weight 2 is a differential on $X(G)$: since $d(\gamma \cdot z) = (cz + d)^{-2}dz$, we have

$$\gamma^*(f(z) \, dz) := f(\gamma \cdot z) \, d(\gamma \cdot z) = (f |_2 \gamma)(z) \, dz = f(z) \, dz.$$  

Forms of higher (even) weights $2k$ are sections of appropriate line bundles on $X(G)$ ($k$-fold differentials).
A modular form for $G$ is a holomorphic modular function on $\mathfrak{h}^*$. Let $M_k(G)$ be the $\mathbb{C}$-vector space of modular forms of weight $k$ for $G$.

**Theorem**. $\dim_{\mathbb{C}} M_k(G) < +\infty$.

For instance, for $G = \Gamma_0(N)$, we have $\dim_{\mathbb{C}} M_k(G) \approx \frac{kN}{12}$. 

Cusp forms, $L$-series

Assume for the moment that $G = \Gamma_0(N)$: the definitions implies that $f \in M_k(G)$ satisfies $f(z + 1) = f(z)$ and has a Fourier expansion at infinity $f(z) = \sum_{n \geq 0} a_n q^n$, where $q = \exp(2i\pi z)$. (For a general congruence subgroup and a general cusp, there is an expansion in $q^{1/H}$, depending on the width $H \geq 1$ of the cusp, for an appropriate local parameter $q$.)

Let $S_k(G) \subset M_k(G)$ be the subspace of cusp forms: vanishing at all cusps. In particular, $a_0 = 0$ and we can define associated $L$-series, for $f \in S_k(G)$:

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}, \quad L(f, \chi, s) = \sum_{n \geq 1} a_n \chi(n) n^{-s},$$

where $\chi$ is a Dirichlet character. The $a_n = O(n^C)$ are polynomially bounded $\Rightarrow$ those functions are in principle defined for $\Re s$ big enough, in a right half-plane. In fact, they are entire functions. Completing them by a gamma factor, we obtain $\Lambda(f, s)$ satisfying a functional equation relating $s$ to $k - s$. Critical values $L(f, j)$, for integers $0 < j < k$, are of particular interest.
Hecke operators, Atkin-Lehner theory (1/2)

Still assume that $G = \Gamma_0(N)$. (Analogous results hold for $\Gamma_1(N)$.) There is a canonical decomposition

$$S_k(G) = S_k(G)_{\text{old}} \oplus S_k(G)_{\text{new}},$$

where $S_{\text{old}}$ contains the forms from $S_k(\Gamma_0(M))$, $M$ a strict divisor of $N$; and $S_{\text{new}}$ is the interesting part. (A basis of $S_{\text{new}}$ can be computed via the intersection of kernels of explicit linear operators associated to divisors of $N$.)

For any integer $n \geq 1$ we have a Hecke operator $T_n$ on $M_k(G)$. These linear “averaging” operators commute and satisfy nice multiplicativity relations, e.g. $T_{mn} = T_mT_n$ when $(m, n) = 1$ and $(mn, N) = 1$, or formulas expressing $T_{p^i}$ in terms of $T_p$ for $p$ prime. Formally,

$$T_n f := \sum_{\gamma \in \Gamma_0(N) \setminus D_n} f |_{k \gamma},$$

where $D_n$ is the set of matrices of determinant $n$ in $M_2(\mathbb{Z})/\{ -\text{Id}, \text{Id} \}$. (The sum is finite. We extend the action $f |_{k \gamma}$ from $\text{PSL}_2(\mathbb{Z})$ to $D_n$ by multiplying our formula for $\gamma \in \text{PSL}_2(\mathbb{Z})$ by $n^{k-1}$.)
Hecke operators, Atkin-Lehner theory (2/2)

Nice properties of Hecke operators:

- all $T_n$ with $(n, N) = 1$ are diagonalizable, their eigenvalues are algebraic integers;
- they stabilize $S_k(G)$, in fact both $S_{\text{new}}$ and $S_{\text{old}}$ separately;
- there exist a $\mathbb{C}$-basis of $S_{\text{new}}$ of simultaneous eigenvectors for all $T_n$;
- if $f = \sum_{n \geq 1} a_n q^n \in S_{\text{new}}$ is an eigenvector for all $T_n$, then $a_1 \neq 0$ and we can normalize $f$ so that $a_1 = 1$; then $T_n f = a_n f$. Such a form is called primitive.

- a primitive form satisfies a product formula

$$L(f, s) = \prod_{p|N} (1 - a_p p^{-s})^{-1} \prod_{p \nmid N} \left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1}.$$  

- if $f = \sum a_n q^n$ is primitive, then $\mathbb{Q}(f) := \mathbb{Q}(a_2, a_3, \ldots)$ is a number field.
Example

The simplest example is

\[ \Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 \ldots, \]

the only primitive form in \( S_{12}(\text{PSL}_2(\mathbb{Z})) \).

Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \) and \( L(E, s) = \sum_{n \geq 1} a_n n^{-s} \) its \( L \)-series. Then

\[ \sum_{n \geq 1} a_n q^n \in S_2(\Gamma_0(N))_{\text{new}} \text{ is primitive.} \]

For instance, let

\[ E : y^2 + y = x^3 - x^2 - 10x - 20 \quad (=11a1), \]

then the corresponding primitive form is

\[ q \prod_{n \geq 1} (1 - q^n)^2(1 - q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 \ldots. \]
Periods and critical $L$-values (1/2)

Let $f \in S_k(G)$, the period

$$\int_r^s f(z)z^j \, dz, \quad j \in \mathbb{Z}_{\geq 0},$$

is well-defined for any $r, s \in \mathbb{P}^1(\mathbb{Q})$. (The integral does not depend on the path in $\mathfrak{h}$ joining the cusps since $f$ is holomorphic in $\mathfrak{h}$, and it converges since $f$ decreases exponentially at cusps.)

Let $f = \sum_n a_n q^n \in S_2(G)$; heuristically, periods should be related to $L$-values, barring convergence issues...

$$2i\pi \int_{i\infty}^0 f(z)z^j \, dz \approx \sum_n a_n \int_{i\infty}^0 2i\pi \exp(2i\pi nz)z^j \, dz \approx \frac{j!}{(-2i\pi)^j} L(f, j + 1).$$

$$= (-2i\pi n)^{-j} \cdot \frac{1}{n} \cdot \Gamma(j+1)$$
It can actually be proven rigorously in a more general form:

**Theorem.** Let \( f \in S_k(G') \), \( G \) a congruence subgroup. Then

\[
2i\pi \int_{i\infty}^{0} f(z)z^j \, dz = \frac{j!}{(-2i\pi)^j} L(f, j + 1),
\]

for all critical \( 0 \leq j \leq k - 2 \).

Similarly for twists by a primitive Dirichlet character of conductor \( D > 1 \), in weight 2:

\[
\frac{\tau(\chi)}{D} \sum_{a \mod D} \overline{\chi(a)} 2i\pi \int_{i\infty}^{-a/D} f(z) \, dz = L(f, \chi, 1),
\]

as well as more complicated generalizations in higher weight.

*Periods know all about (twisted) critical \( L \)-values.*
Complex modular symbols, weight 2

Let $\Delta_0 := \text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$: given $s, r \in \mathbb{P}^1(\mathbb{Q})$, think of the divisor $[s] - [r]$, as an oriented path in $\mathbb{H}$ connecting $r \to s$. E.g., the semicircle connecting $s$ to $r$, or a vertical line through $r$ if $s = i\infty$. Those divisors generate $\Delta_0$. Note that $\Delta_0$ is a $\text{GL}_2(\mathbb{Q})$-module: for $g \in \text{GL}_2(\mathbb{Q})$,

$$g \cdot ([s] - [r]) := [g \cdot s] - [g \cdot r].$$

In matrix form, if $r = (a : c), s = (b : d)$, the matrix $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ codes the path $[s] - [r]$; then the path $g \cdot ([s] - [r])$ is identified with the matrix $g \times \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$.

Let $f \in S_2(G)$, we define a map $\psi_f$ from $\Delta_0$ to $\mathbb{C}$ by

$$[s] - [r] \mapsto 2i\pi \int_r^s f(z) \, dz$$

(Well-defined: Chasles relation.) Since $f \in S_2(G)$, we have

$$\int_{\gamma \cdot r}^{\gamma \cdot s} f(z) \, dz = \int_{\gamma \cdot r}^{\gamma \cdot s} f(\gamma \cdot z) \, d(\gamma \cdot z) = \int_r^s f(z) \, dz.$$

Thus $\psi_f \in \text{Hom}_G(\Delta_0, \mathbb{C})$: $\psi(\gamma \cdot D) = \psi(D)$ for all $\gamma \in G$. 

The relevant period integrals attached to $f \in S_k(G)$ are the

$$
\int_r^s f(z) z^j \, dz, \quad 0 \leq j \leq k - 2.
$$

Let $V := \text{Sym}^{k-2}(\mathbb{C}^2)$, realized as the space of homogeneous polynomials of degree $k - 2$ in $\mathbb{C}[X, Y]$, together with the right $\text{SL}_2(\mathbb{Z})$ action: $(P \mid \gamma)(X, Y) := P((X, Y) \times \gamma^{-1})$.

There is a natural right action on $\text{Hom}(\Delta_0, V)$: for $\phi \in \text{Hom}(\Delta_0, V)$, define $\phi \mid \gamma$ by

$$(\phi \mid \gamma)(D) := \phi(\gamma \cdot D) \mid \gamma, \quad \forall D \in \Delta_0.$$ 

Define $\psi_f \in \text{Hom}(\Delta_0, V)$ by

$$
\psi_f([s] - [r]) := 2i\pi \int_r^s f(z)(zX + Y)^{k-2} \, dz \in V.
$$

Then $\psi_f \mid \gamma = \psi_f$ for any $\gamma \in G$! Again, $\psi_f \in \text{Hom}_G(\Delta_0, V)$. 

**Complex modular symbols, general weight $k$**
Proof of $\psi_f \in \text{Hom}_G(\Delta_0, V)$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Recall that

$$f |_k \gamma = f,$$

$$(P | \gamma)(X, Y) := P((X, Y) \times \gamma^{-1}), \quad P \in V,$$

$$(\phi | \gamma)(D) := \phi(\gamma \cdot D) | \gamma,$$

$$\psi_f([s] - [r]) := 2i\pi \int_r^s f(z)(zX + Y)^{k-2} \, dz \in V.$$

We have

$$\psi_f([s] - [r]) | \gamma^{-1} = 2i\pi \int_r^s f(z)\left((X, Y)\gamma\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)\right)^{k-2} \, dz$$

$$= 2i\pi \int_r^s f(\gamma \cdot z)/(cz + d)^k\left((X, Y)\left(\begin{smallmatrix} az + b \\ cz + d \end{smallmatrix}\right)\right)^{k-2} \, dz$$

$$= 2i\pi \int_r^s f(\gamma \cdot z)\left((X, Y)\left(\begin{smallmatrix} \gamma \cdot z \\ 1 \end{smallmatrix}\right)\right)^{k-2} d(\gamma \cdot z)$$

$$= 2i\pi \int_{\gamma \cdot r}^{\gamma \cdot s} f(z)\left((X, Y)\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)\right)^{k-2} \, dz = \psi_f(\gamma \cdot ([s] - [r])) \quad \square$$
Cohomological interpretation

Let $G \subset \text{PSL}(2, \mathbb{Z})$ be a congruence subgroup, and $V$ be a right $G$-module. One defines the cohomology of the modular curve $X(G)$ with coefficients in $V$, the group of interest being $H^1_c(X(G), V)$; one can again define Hecke operators in this context.

**Back to previous example:** $G = \Gamma_0(N), \ V = \text{Sym}^{k-2} \mathbb{C}^2,$

\[(P \mid \gamma)(X, Y) = P((X, Y)\gamma^{-1}), \quad P \in V.\]

We recover classical $\mathbb{C}$-vector spaces of holomorphic modular forms for $G$:

**Theorem (Eichler-Shimura).**

\[H^1_c(X(G), V) \cong_{\text{Hecke}} S_k(G) \oplus M_k(G)\]

Cohomology classes are not that explicit...
Abstract modular symbols (1/3)

Classical modular symbols for $G = \Gamma_0(N)$ provide

- an algebraic version of periods of holomorphic forms,
- a way to describe (and compute!) $M_k(G)$ as a Hecke-module from finite rational data,

For general $G$ (congruence subgroup) and $V$ (over $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Q}_p$, $\mathbb{Z}$, infinite dimensional...), they also are

- a concrete realization of cohomology classes $H^1_c(X(G), V)$ that afford a painless way to define (and compute!) general spaces of “modular forms”, or rather systems of Hecke eigenvalues, using basic linear algebra.
Let $\Delta_0 := \text{Div}^0(P^1(Q))$, generated by the divisors $[\beta] - [\alpha]$, which we denote by $\{\alpha, \beta\}$ and see as a path through the completed upper half plane $\mathfrak{h}^*$ linking the two cusps $\alpha \to \beta$. This is a left $\text{GL}(2, \mathbb{Q})$-module via fractional linear transformations:

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [(u : v)] := [(au + bv : cu + dv)].
$$

Let $G \subset \text{PSL}_2(\mathbb{Z})$ be a subgroup of finite index and let $V$ be a right $G$-module. $\text{Hom}(\Delta_0, V)$ becomes a right $G$-module via

$$(\phi | \gamma)(D) := \phi(\gamma \cdot D) | \gamma$$

We define the $V$-valued modular symbols on $G$ by

$$\text{Symb}_G(V) := \text{Hom}_G(\Delta_0, V), \quad \phi | \gamma = \phi, \forall \phi \in G.$$
Abstract modular symbols (3/3)

**Theorem** (Ash-Stevens). Let $G$ be a congruence subgroup and $V$ a right $G$-module. Provided that the orders of torsion elements of $G$ act invertibly on $V$ (e.g. if $V$ is a vector space), we have a canonical isomorphism

$$\text{Symb}_G(V) \cong H^1_c(X(G), V).$$

Assume $V$ also allows a right action by the semi-group $\text{GL}(2, \mathbb{Q}) \cap M_2(\mathbb{Z})$, then we can define a Hecke action on $\text{Symb}_G(V)$. E.g. if $G = \Gamma_0(N)$ and $\ell$ is prime, then

$$T_\ell \phi := \phi \mid (\ell \, 0 \, 0 \, 1) + \sum_{a=0}^{\ell-1} \phi \mid (1 \, a \, \ell \, 0).$$

If $\sigma := \left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ normalizes $G$, then it acts as an involution on $\text{Symb}_G(V)$; if 2 acts invertibly on $V$, this yields a decomposition

$$\text{Symb}_G(V) = \text{Symb}_G(V)^+ \oplus \text{Symb}_G(V)^-$$

into eigenspaces for this action.
Computing $\Delta_0$ as a $\mathbb{Z}[G]$-module (1/5)

Let $G \subset \Gamma = \text{PSL}(2, \mathbb{Z})$ and $B = [\Gamma : G] < +\infty$. The subgroup $G$ is given via an enumeration $(m_1, \ldots, m_B)$ of matrices representing $G \backslash \text{PSL}(2, \mathbb{Z})$. Assume that

- the coset representatives $m_i$ have size $O(\log B)^C$,
- the map $(\gamma \in \Gamma \mapsto$ its coset), i.e. the unique $i$ such that $G\gamma = Gm_i$, is computed in polynomial time $O(\log \|\gamma\| + \log B)^C$.

In particular, both the membership problem ($\gamma \in G$?) and test for equivalence ($\gamma_1 \sim_G \gamma_2$?) are solved in polynomial time in the size of the $\gamma_i \in \Gamma$.

**Theorem** (Manin). If $B = 1$ ($G = \Gamma$), then

$$
\Delta_0 \simeq_\Gamma \mathbb{Z}[\Gamma]/I, \quad \text{where} \quad I := \mathbb{Z}[\Gamma](1 + \sigma) + \mathbb{Z}[\Gamma](1 + \tau + \tau^2),
$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\Gamma = \langle \sigma, \tau \rangle$.

In this case a $V$-valued modular symbol $\phi \in \text{Hom}_\Gamma(\Delta_0, V)$ is defined by $v_\sigma, v_\tau \in V$ s.t.

$$
v_\sigma \mid (1 + \sigma) = v_\tau \mid (1 + \tau + \tau^2) = 0.
$$
Computing $\Delta_0$ as a $\mathbb{Z}[G]$-module (2/5)

In principle, Manin’s theorem yields a presentation of $\Delta_0$ as a $\mathbb{Z}[G]$ module: $\mathbb{Z}[\Gamma]$ is free (generated by the $m_i$), and quotienting out yields relations of the form

$$m_i(1 + \sigma) = m_i + m_i\sigma = m_i + \gamma_{i,j}m_j \in I,$$

for some $j$ and $\gamma_{i,j} \in G$. There’s a neater, simpler, way.

**Fact:** the torsion elements in $\text{PSL}_2(\mathbb{Z})$ have order 2 or 3.

**Theorem** (Pollack-Stevens). Let $G \subset \text{PSL}_2(\mathbb{Z})$ be a subgroup of finite index $B$ without 3-torsion. There exist a connected fundamental domain $F$ for the action of $G$ on $\mathfrak{h}^*$ all of whose vertices are cusps and whose boundary is a union of unimodular paths.
Computing $\Delta_0$ as a $\mathbb{Z}[G]$-module (3/5)

Proof. Start from the hyperbolic triangle $R = (0, 1, i\infty)$, a fundamental domain for $\Gamma_0(2)$. We use Farey dissection to add further triangles until we obtain the full domain: given 2 cusps $(a : b) < (c : d)$ on the boundary of current domain

$$\alpha_1 R \cup \cdots \cup \alpha_r R,$$

such that $ad - bc = 1$, the third vertex of the triangle $\left(\begin{array}{cc} a & c \\ b & d \end{array}\right) R$ is the mediant $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Add the new triangle to the domain if and only if $\alpha_i \tau^j \left(\begin{array}{cc} a & c \\ b & d \end{array}\right)^{-1} \notin \Gamma$, $\forall 1 \leq i \leq r$, $0 \leq j \leq 2$. The algorithm stops after at most $B$ triangles are added. \hfill \Box

If $G$ has 3-torsion: essentially the same, but we must split triangles in 3: $R = T \cup \tau T \cup \tau^2 T$, where $T = (0, e^{i\pi/3}, i\infty)$, and we sometimes add only $1/3$ of a triangle ($\alpha T$ instead of $\alpha R$).

Theorem. Under our assumptions on $G$, the fundamental domain $F$ can be computed in time $\tilde{O}(B)$.

N.B. some complexity estimates only depend on the number of cusps rather than $B$, which is advantageous: $G = \Gamma_0(p)$ has index $p + 1$ but only 2 cusps.
Computing $\Delta_0$ as a $\mathbb{Z}[G]$-module (4/5)

- If $G$ has no torsion then $\Delta_0$ is generated by the $g_i := [c_{i+1}] - [c_i]$, paths between consecutive vertices of $F$, with the single relation $\sum_i g_i = 0$!

- If $G$ has 2-torsion, then it can happen that $\gamma_i g_i = -g_i$ for some $\gamma_i \in G$ swapping $c_i$ and $c_{i+1}$ (implies $\gamma_i$ has order 2). Then $(1 + \gamma_i) \cdot g_i = 0$ and $g_i$ is torsion.

- If $G$ has 3-torsion, then we have extra torsion relations corresponding to going around a triangle $\alpha R$ fixed by an element of order 3.
Computing $\Delta_0$ as a $\mathbb{Z}[G]$-module (5/5)

Summary: In general, we obtain

- a “minimal” system of generators $(g_i), i \leq n, g_n = [\infty] - [0]$.
- relations explicitly written down (without computation):
  - one relation for each conjugacy class of 2-torsion elements in G: $(1 + \gamma_i) \cdot g_i = 0, 1 \leq i \leq s$
  - one for each pairs of conjugacy classes of 3-torsion elements: $(1 + \gamma_i + \gamma_i^2) \cdot g_i = 0, s + 1 \leq i \leq s + r$.
  - and one “boundary relation” (walk around the fundamental domain and come back to starting point).

Corollary. Given G a finite index subgroup and V a right G-module. Choose any $n - 1$ elements $v_i \in V$, compatible with the torsion relations when $i \leq s + r$ (e.g. $v_i(1 + \gamma_i) = 0$, i.e restrict $v_i$ to an eigenspace $V_i \subset V$). Solve for $v_n$ so that the boundary relation is satisfied. Then $\phi(g_i) = v_i$ uniquely defines a modular symbol $\phi$, and all modular symbols arise in this way.
Recall that a (non-trivial) path \((a : c) \rightarrow (b : d)\) is encoded by the matrix \((\begin{array}{cc} a & b \\ c & d \end{array}) \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})^+\). A unimodular path has determinant 1.

Recall that the subgroup \(G\) is given via an enumeration \((m_1, \ldots, m_B)\) of matrices representing \(G \setminus \text{PSL}(2, \mathbb{Z})\).

- the discrete logs \(m_i = \sum \lambda_{i,j} g_j, i \leq B\), are precomputed: \(\widetilde{O}(B^2)\) time and space.
- a path \(\infty \rightarrow (b : d)\) can be written as a sum of \(O(\log \max(|b|, |d|))\) unimodular paths.
  
  Proof: write the finite continued fraction of \(b/d\). The successive convergents satisfy 
  
  \[(p_{-1} : q_{-1}) = (1 : 0), \ldots, (p_n : q_n) = (b : d)\] and 
  
  \[\det \left( \begin{array}{cc} p_i & p_{i+1} \\ q_i & q_{i+1} \end{array} \right) = \pm 1.\]

- a path \((a : c) \rightarrow (b : d)\) can be written as a sum of \(O(\log \max(|a|, |b|, |c|, |d|))\) unimodular paths. Proof: \((a : c) \rightarrow (1 : 0) \rightarrow (b : d)\). Better (halve number of paths on average), \(U^{-1} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & b' \\ 0 & d' \end{array} \right)\) (HNF), then \(U \cdot \gamma_i\).

- a unimodular path is uniquely written as \(\gamma \cdot m_i\) for some \(\gamma \in G\).
Let $f \in S_k(G), V = \mathbb{C}[X, Y]_{k-2}$. Recall that $\psi_f \in \text{Symb}_G(V)$ defined by

$$\psi_f([s] - [r]) := 2i\pi \int_r^s f(z)(zX + Y)^{k-2} \, dz \in V$$

knows about critical $L$-values:

$$\psi_f([0] - [i\infty]) = \sum_{0 \leq j \leq k-2} X^j Y^{k-2-j} \binom{k-2}{j} \frac{j!}{(-2i\pi)^j} L(f, j + 1).$$

**Theorem** (Manin, Shimura). There exist $\Omega_{\pm f} \in \mathbb{C}$ such that

$$\frac{L(f, \chi, j + 1)}{(-2i\pi)^j} \in \Omega_{\pm f} \mathbb{Q},$$

for all Dirichlet characters $\chi$ and $j \leq k - 2$. (Precisely in $\Omega_{f}^{(-1)^j} \chi(-1) \mathbb{Q}$.)

By fixing an embedding of $\mathbb{Q} \subset \overline{\mathbb{Q}}_p$, we can consider those renormalized $L(f, \chi, j + 1)$ as $p$-adic numbers!
Fix a prime $p$. Let $\Gamma$ be a congruence subgroup of level prime to $p$ and $G := \Gamma \cap \Gamma_0(p)$. Let $f \in S_k(G)$ be a normalized eigenform, with $T_p f = \alpha f$.

The $p$-adic $L$-function $\mu_f$ associated to $f$ should be a way to associate $(j, \chi) \mapsto \mathcal{L}(f, \chi, j + 1)$. It’s going to be a $p$-adic distribution, mapping “nice functions” (characters, polynomials) to $p$-adic numbers. Precisely, assume that $v_p(\alpha) < k - 1$; for any finite order character $\chi$ of $\mathbb{Z}_p^\times$ of conductor $p^n$ and any integer $0 \leq j \leq k - 2$, we want

$$
\mu_f(z^j \cdot \chi) := \alpha^{-n} p^{n(j+1)} \frac{j!}{\tau(\chi^{-1})} \mathcal{L}(f, \chi^{-1}, j + 1) \in \overline{\mathbb{Q}_p}.
$$

This defines $\mu_f$ uniquely, for a given choice of complex periods $\Omega_f^\pm$. The distribution $\mu_f$ can be evaluated on locally analytic functions ($\chi$ is locally constant but not analytic!); we write

$$
\int g(t) \, d\mu_f(t) \text{ for } \mu_f(g).
$$

Hard to compute when defined this way: Riemann sums with (at least) $p^n$ terms to evaluate modulo $p^n$. 

**p-adic L functions (2/4)**
Let \( V = \mathcal{D}_k(\mathbb{Z}_p) =: \mathcal{D} \), the space of locally analytic \( p \)-adic distributions on \( \mathbb{Z}_p \), with weight \( k - 2 \) action of \( G \):

\[
(\mu |_{k} \gamma)(g) := \mu(\gamma \cdot g), \quad \text{where} \quad (\gamma \cdot g)(z) := (a + cz)^{k-2} f \left( \frac{b + dz}{a + cz} \right).
\]

This defines \( \text{Symb}_G(\mathcal{D}) \), the space of overconvergent modular symbols.

Composing with the \( p \)-adic period map \( \rho_k : \mathcal{D} \to \text{Sym}^{k-2} \mathbb{Q}_p^2 \), given by

\[
\mu \mapsto \int (Y - tX)^{k-2} d\mu(t),
\]

defines specializations

\[
\text{Symb}_G(\mathcal{D}) \to \text{Symb}_G(\text{Sym}^{k-2} \mathbb{Q}_p^2).
\]

The target of this map is finite dimensional while the source has infinite dimension!

Nevertheless, by restricting to natural subspaces, Pollack and Stevens obtain a Hecke-equivariant isomorphism.
The $p$-adic slope of a primitive form $f \in S_k(G)$ is $v_p(a_p)$, it is $\leq k - 1$. *(Critical slope when equality.)*

**Theorem** (Stevens). The map

$$\text{Symb}_G(\mathcal{D})^{(<k-1)} \to \text{Symb}_G(\text{Sym}^{k-2} \mathbb{Q}_p)^{(<k-1)}$$

is an isomorphism, compatible with Hecke action.

**Theorem** . Let $f$ be primitive for $G$ of non-critical slope and $\phi_f \in \text{Symb}_G(\text{Sym}^{k-2} \mathbb{Q}_p)$ be the corresponding classical modular eigensymbol. Let $\Phi_f$ be the unique overconvergent eigensymbol lifting $\phi_f$. Then $\Phi_f([0] - [i\infty])$ is the $p$-adic $L$-function of $g$.

The case of critical slope can also be dealt with in a similar way.

**Theorem** . The $\Phi_f([0] - [i\infty])(z^j)$ modulo $p^{M-j}$, $j \leq M$ can be computed in time $p^{M^{O(1)}}$: polynomial time for fixed $p$. 