

# CM-Points on Straight Lines

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Bordeaux

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## Complex Multiplication

Lattices

$j$ -invariant

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Class Field Theory

The Class Number

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Special Points and Special Curves

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Kühne's "uniformity observation"

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## The Proof

Equality of CM-fields

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Discriminants with Class Group Annihilated by 2

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- ▶  $\{\text{lattices up to isomorphism}\} = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$

$\mathbb{H} = \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$  “Poincaré (half)plane”

# $j$ -invariant

- ▶  **$j$ -invariant:**  $SL_2(\mathbb{Z})$ -automorphic function on  $\mathbb{H}$  satisfying
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$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

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 $|q|$  small when  $\mathrm{Im} \tau$  large  $\implies |j(\tau)|$  large when  $\mathrm{Im} \tau$  large
- ▶  $j$ -invariant “classifies lattices”:
$$\langle \tau, 1 \rangle \cong \langle \tau', 1 \rangle \iff j(\tau) = j(\tau')$$

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  - ▶ if  $\tau$  is root of  $at^2 + bt + c \in \mathbb{Z}[t]$ ,  $(a, b, c) = 1$  then  $\Delta = b^2 - 4ac$  and  $\tau = \frac{-b + \sqrt{\Delta}}{2a}$ .

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- ▶  $h(\Delta)$  the class number of the order  $\mathcal{O}_\Delta$
- ▶ moreover:  $\text{Gal}(K(j(\tau))/K) = \text{Cl}(\Delta)$

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- ▶ In particular (Heegner-Stark) there exist thirteen  $\Delta$  with  $h(\Delta) = 1$  (the corresponding  $j$  belong to  $\mathbb{Z}$ ):

$\Delta$	$-3$	$-3 \cdot 2^2$	$-3 \cdot 3^3$	$-4$	$-4 \cdot 2^2$	$-7$	$-7 \cdot 2^2$	$-8$
$j$	$0$	$2^4 3^3 5^3$	$-2^{15} 3 \cdot 5^3$	$2^6 3^3$	$2^3 3^3 11^3$	$-3^3 5^3$	$3^3 5^3 17^3$	$2^6 5^3$
$\Delta$	$-11$	$-19$	$-43$	$-67$	$-163$			
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$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007 \dots$$



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$\Delta$	$-3$	$-3 \cdot 2^2$	$-3 \cdot 3^3$	$-4$	$-4 \cdot 2^2$	$-7$	$-7 \cdot 2^2$	$-8$
$j$	$0$	$2^4 3^3 5^3$	$-2^{15} 3 \cdot 5^3$	$2^6 3^3$	$2^3 3^3 11^3$	$-3^3 5^3$	$3^3 5^3 17^3$	$2^6 5^3$
$\Delta$	$-11$	$-19$	$-43$	$-67$	$-163$			
$j$	$-2^{15}$	$-2^{15} 3^3$	$-2^{18} 3^3 5^3$	$-2^{15} 3^3 5^3 11^3$	$-2^{18} 3^3 5^3 23^3 29^3$			

- ▶ A funny example (Hermite):

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007 \dots$$

- ▶  $\tau = \frac{1 + \sqrt{-163}}{2}$

- ▶  $e^{\pi\sqrt{163}} = -e^{2\pi i\tau} \approx -j(\tau) + 744 \in \mathbb{Z}$

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- ▶  $h(\Delta) \rightarrow \infty$  as  $|\Delta| \rightarrow \infty$  (Siegel)
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- ▶ Currently all  $\Delta$  with  $h_{\Delta} \leq 100$  are known (Watkins 2006).

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## Polynomials $\Phi_N, N \leq 3$

$$\Phi_1(x, y) = x - y$$

$$\begin{aligned}\Phi_2(x, y) = & -x^2y^2 + x^3 + y^3 + 1488x^2y + 1488xy^2 + 40773375xy \\ & - 162000x^2 - 162000y^2 + 8748000000x + 8748000000y - 15746400000000\end{aligned}$$

$$\begin{aligned}\Phi_3(x, y) = & x^4 + y^4 - x^3y^3 + 2232x^3y^2 + 2232x^2y^3 - 1069956x^3y - 1069956xy^3 \\ & + 36864000x^3 + 36864000y^3 + 2587918086x^2y^2 \\ & + 8900222976000x^2y + 8900222976000xy^2 + 452984832000000x^2 + 452984832000000y^2 \\ & - 770845966336000000xy + 1855425871872000000000x + 1855425871872000000000y\end{aligned}$$



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- ▶ In particular: all CM-points belonging to non-special straight lines defined over  $\mathbb{Q}$  can (in principle) be listed explicitly.
- ▶ Bajolet (2014): software to determine all CM-points on a given line.

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**Theorem** (A., B., Pizarro; May 2014) If a CM-points belongs to a non-special straight line over  $\mathbb{Q}$  then we have one of the two cases above.

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Field $L$	$\Delta$	$\text{Cl}(\Delta)$
$\mathbb{Q}$	$-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163$	trivial
$\mathbb{Q}(\sqrt{2})$	$-24, -32, -64, -88$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Q}(\sqrt{3})$	$-36, -48$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Q}(\sqrt{5})$	$-15, -20, -35, -40, -60, -75, -100, -115, -235$	$\mathbb{Z}/2\mathbb{Z}$
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$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	$-96, -192, -288$	$(\mathbb{Z}/2\mathbb{Z})^2$
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- ▶ If  $\mathbb{Q}(\tau_1) = \mathbb{Q}(\tau_2)$  then  $\Delta_1/\Delta_2 \in \{1, 4, 1/4\}$  or  $\Delta_1, \Delta_2 \in \{-3, -12, -27\}$ .



# The Proof

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- ▶  $[L : \mathbb{Q}] = h(\Delta_1) = h(\Delta_2) \geq 3$
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- ▶  $\tau_1 = \frac{-b_1 + \sqrt{\Delta}}{2}$ ,  $\tau_2 = \frac{-b_2 + \sqrt{\Delta}}{2}$  or  $\tau_2 = \frac{-b_2 + 2\sqrt{\Delta}}{2}$
- ▶ In the first case  $j(\tau_1) = j(\tau_2)$
- ▶ In the second case  $(j(\tau_1), j(\tau_2)) \in Y_0(2)$ .

## The Proof (continued)

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One rules them out using PARI.



## Complex Multiplication

Lattices

$j$ -invariant

Complex Multiplication

Class Field Theory

The Class Number

## Theorem of André

Special Points and Special Curves

Theorem of André

## CM-Points on Straight Lines

Kühne's "uniformity observation"

CM-Points on Straight Lines

## The Proof

Equality of CM-fields

The Proof

## Proof of Theorem ECMF

Discriminants with Class Group Annihilated by 2

Proof of Theorem ECMF

# Discriminants with Class Group Annihilated by 2

## Known $\Delta$ with $\text{Cl}(\Delta)^2 = 1$

- 3,  $-3 \cdot 2^2$ ,  $-3 \cdot 3^2$ ,  $-3 \cdot 4^2$ ,  $-3 \cdot 5^2$ ,  $-3 \cdot 7^2$ ,  $-3 \cdot 8^2$ , -4,  $-4 \cdot 2^2$ ,  $-4 \cdot 3^2$ ,  $-4 \cdot 4^2$ ,  $-4 \cdot 5^2$ ,  
- 7,  $-7 \cdot 2^2$ ,  $-7 \cdot 4^2$ ,  $-7 \cdot 8^2$ , -8,  $-8 \cdot 2^2$ ,  $-8 \cdot 3^2$ ,  $-8 \cdot 6^2$ , -11,  $-11 \cdot 3^2$ ,  
- 15,  $-15 \cdot 2^2$ ,  $-15 \cdot 4^2$ ,  $-15 \cdot 8^2$ , -19, -20,  $-20 \cdot 3^2$ , -24,  $-24 \cdot 2^2$ , -35,  $-35 \cdot 3^2$ , -40,  $-40 \cdot 2^2$ ,  
- 43, -51, -52, -67, -84, -88,  $-88 \cdot 2^2$ , -91, -115, -120,  $-120 \cdot 2^2$ , -123, -132, -148, -163,  
- 168,  $-168 \cdot 2^2$ , -187, -195, -228, -232,  $-232 \cdot 2^2$ , -235, -267, -280,  $-280 \cdot 2^2$ , -312,  $-312 \cdot 2^2$ ,  
- 340, -372, -403, -408,  $-408 \cdot 2^2$ , -420, -427, -435, -483, -520,  $-520 \cdot 2^2$ , -532, -555, -595,  
- 627, -660, -708, -715, -760,  $-760 \cdot 2^2$ , -795, -840,  $-840 \cdot 2^2$ , -1012, -1092, -1155,  
- 1320,  $-1320 \cdot 2^2$ , -1380, -1428, -1435, -1540, -1848,  $-1848 \cdot 2^2$ , -1995, -3003, -3315, -5460.

# Discriminants with Class Group Annihilated by 2

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$-3, -3 \cdot 2^2, -3 \cdot 3^2, -3 \cdot 4^2, -3 \cdot 5^2, -3 \cdot 7^2, -3 \cdot 8^2, -4, -4 \cdot 2^2, -4 \cdot 3^2, -4 \cdot 4^2, -4 \cdot 5^2,$   
 $-7, -7 \cdot 2^2, -7 \cdot 4^2, -7 \cdot 8^2, -8, -8 \cdot 2^2, -8 \cdot 3^2, -8 \cdot 6^2, -11, -11 \cdot 3^2,$   
 $-15, -15 \cdot 2^2, -15 \cdot 4^2, -15 \cdot 8^2, -19, -20, -20 \cdot 3^2, -24, -24 \cdot 2^2, -35, -35 \cdot 3^2, -40, -40 \cdot 2^2,$   
 $-43, -51, -52, -67, -84, -88, -88 \cdot 2^2, -91, -115, -120, -120 \cdot 2^2, -123, -132, -148, -163,$   
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► **Weinberger** (1973): All field discriminants  $D$  with  $\text{Cl}(D)^2 = 1$  belong to the list above



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- ▶ **Weinberger** (1973): All field discriminants  $D$  with  $\text{Cl}(D)^2 = 1$  belong to the list above **with at most one exception**.
- ▶ **Corollary:** There exists  $D^*$  such that: if  $\Delta = Df^2$  with  $\text{Cl}(\Delta)^2 = 1$  is **not** in the list then  $D = D^*$ .

# Discriminants with Class Group Annihilated by 2

## Known $\Delta$ with $\text{Cl}(\Delta)^2 = 1$

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- ▶ Class numbers of discriminants from the list are at most 16.

# Discriminants with Class Group Annihilated by 2

## Known $\Delta$ with $\text{Cl}(\Delta)^2 = 1$

- 3,  $-3 \cdot 2^2$ ,  $-3 \cdot 3^2$ ,  $-3 \cdot 4^2$ ,  $-3 \cdot 5^2$ ,  $-3 \cdot 7^2$ ,  $-3 \cdot 8^2$ , -4,  $-4 \cdot 2^2$ ,  $-4 \cdot 3^2$ ,  $-4 \cdot 4^2$ ,  $-4 \cdot 5^2$ ,  
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- ▶ Class numbers of discriminants from the list are at most 16.
- ▶ **Watkins** (2006): the list contains all  $\Delta$  with  $|\text{Cl}(\Delta)^2| = 1$  and  $h(\Delta) \leq 64$ .

# Discriminants with Class Group Annihilated by 2

## Known $\Delta$ with $\text{Cl}(\Delta)^2 = 1$

- 3,  $-3 \cdot 2^2$ ,  $-3 \cdot 3^2$ ,  $-3 \cdot 4^2$ ,  $-3 \cdot 5^2$ ,  $-3 \cdot 7^2$ ,  $-3 \cdot 8^2$ , -4,  $-4 \cdot 2^2$ ,  $-4 \cdot 3^2$ ,  $-4 \cdot 4^2$ ,  $-4 \cdot 5^2$ ,  
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- 1320,  $-1320 \cdot 2^2$ , -1380, -1428, -1435, -1540, -1848,  $-1848 \cdot 2^2$ , -1995, -3003, -3315, -5460.

- ▶ **Weinberger** (1973): All field discriminants  $D$  with  $\text{Cl}(D)^2 = 1$  belong to the list above **with at most one exception**.
- ▶ **Corollary:** There exists  $D^*$  such that: if  $\Delta = Df^2$  with  $\text{Cl}(\Delta)^2 = 1$  is **not** in the list then  $D = D^*$ .
- ▶ Class numbers of discriminants from the list are at most 16.
- ▶ **Watkins** (2006): the list contains all  $\Delta$  with  $|\text{Cl}(\Delta)^2| = 1$  and  $h(\Delta) \leq 64$ .
- ▶ Hence: if  $\Delta$  with  $\text{Cl}(\Delta)^2 = 1$  is **not** in the list then  $h(\Delta) \geq 128$ .

# Proof of Theorem ECMF

- ▶ Assume that  $Q(\tau_1) \neq Q(\tau_2)$  and  $Q(j(\tau_1)) = Q(j(\tau_2))$ .

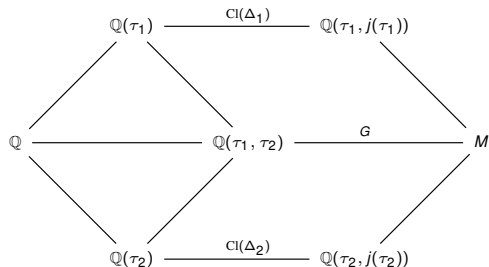
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- ▶ Assume that  $\mathbb{Q}(\tau_1) \neq \mathbb{Q}(\tau_2)$  and  $\mathbb{Q}(j(\tau_1)) = \mathbb{Q}(j(\tau_2))$ .
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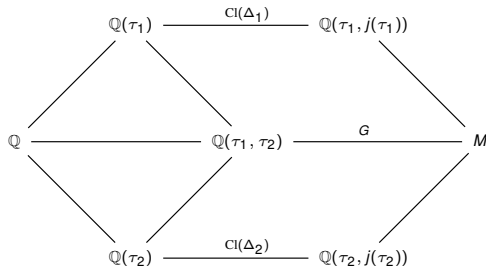
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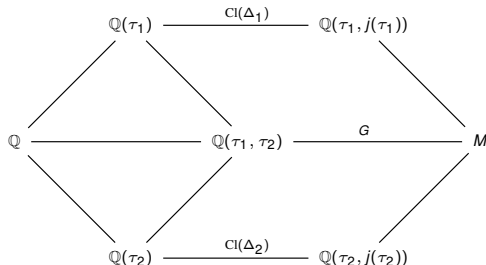
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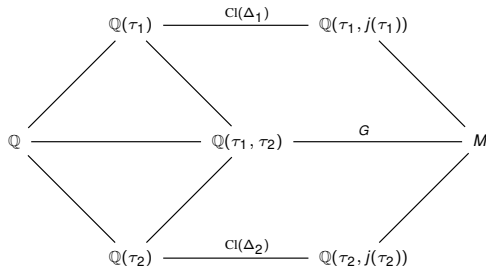
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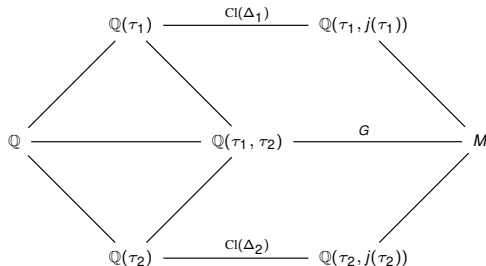
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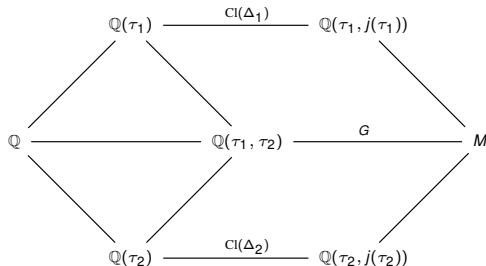
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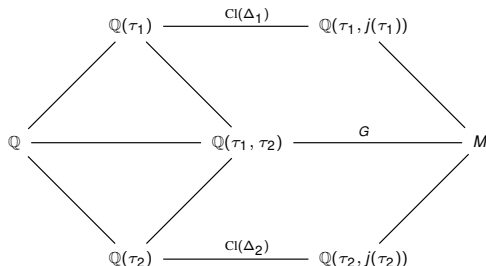
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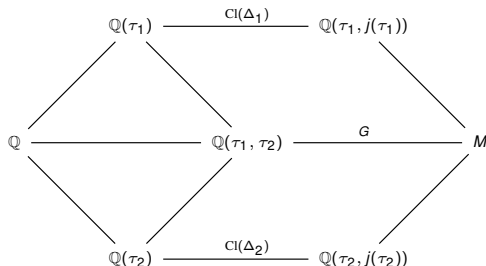
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- ▶ Verification with PARI completes the proof.