Cryptographie à base de courbes elliptiques : algorithmes et implémentation

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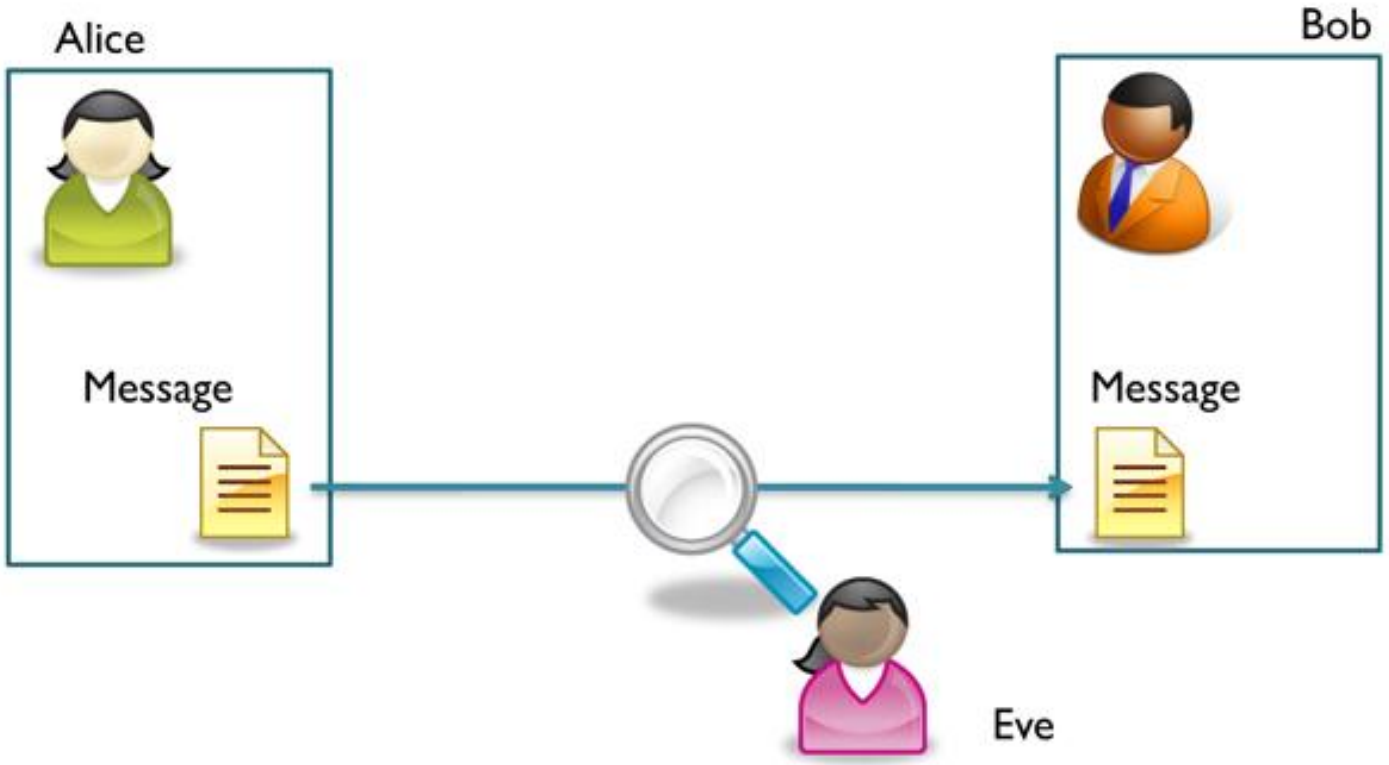
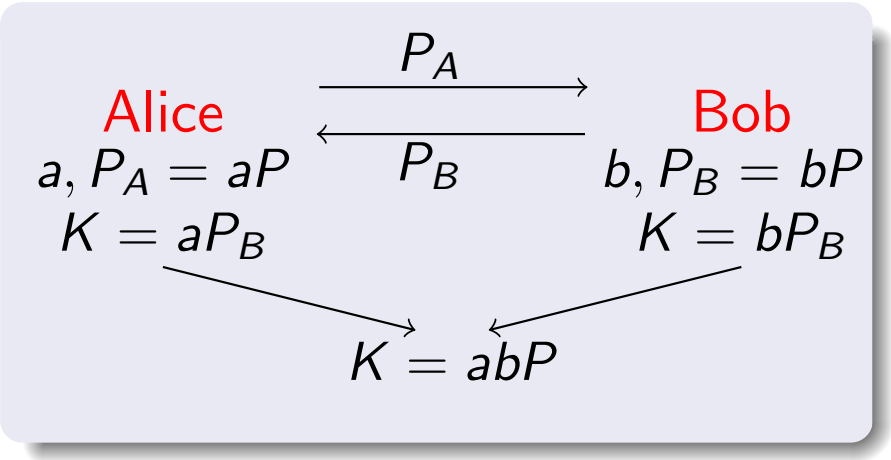
Public key cryptography



Sharing a common secret over an insecure channel

Public key cryptography and groups

- Diffie-Hellman Key Exchange : $(G, +, P)$ public



Security: the Discrete Logarithm Problem (DLP) in G

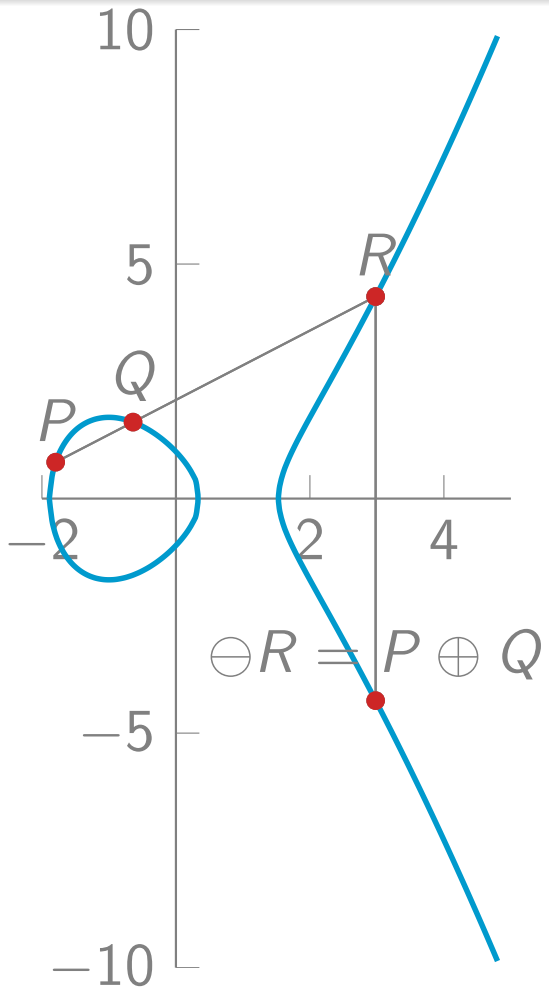
- Given $P, Q \in G$ find (if it exists) λ such that

$$Q = \lambda P$$

Consider $\mathbb{F}_q$, $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$

Weierstrass form

$$y^2 = x^3 + ax + b$$



- Secure implementation : DLP is hard if $r = \#G$ is a large prime number.
- Shorter keys (compared to RSA, group cryptography over finite fields)

Table : Complexity of generic attacks

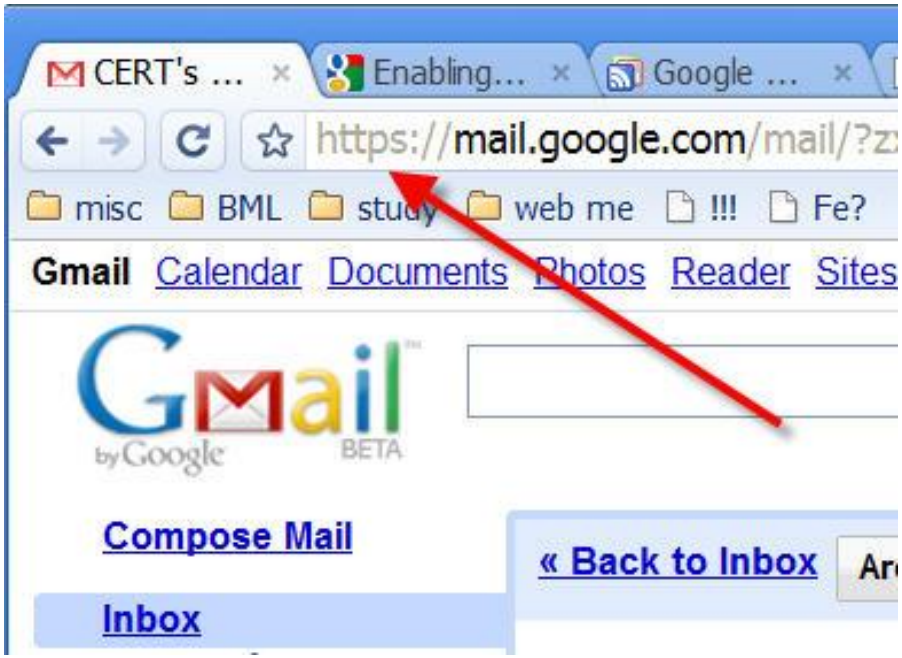
method	Fastest known attack
RSA	Number Field Sieve $exp(\frac{1}{2}(logN)^{\frac{1}{3}}(log \log N)^{\frac{2}{3}})$
ECC	Pollard-rho $\sqrt{r} = exp(\frac{1}{2} \log r)$

Table : Key sizes

Security level	RSA	ECC
80 bits	1024	160
128 bits	3072	256
256 bits	15360	512

ECC in the real world

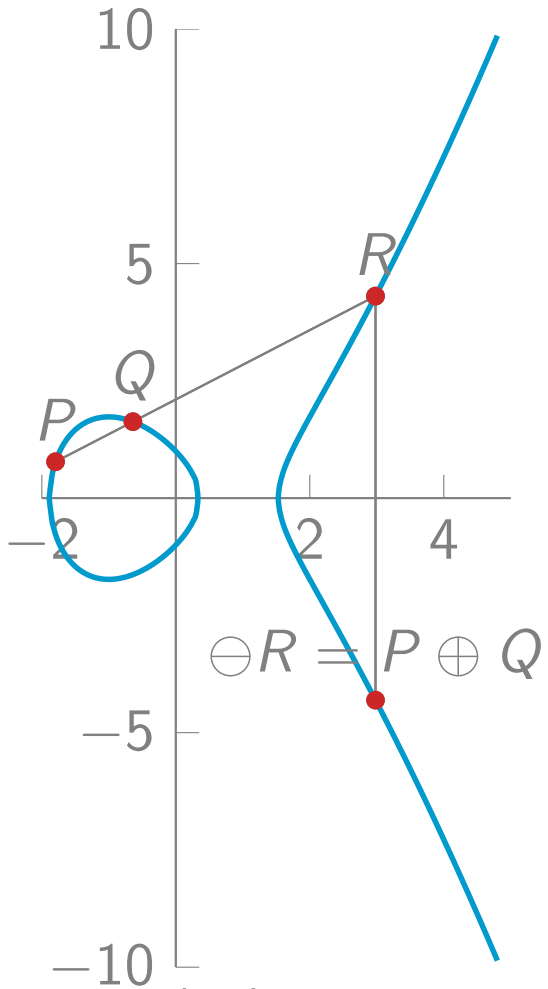
key exchange, signatures, identification



Elliptic versus genus 2 curves

Genus 1 addition

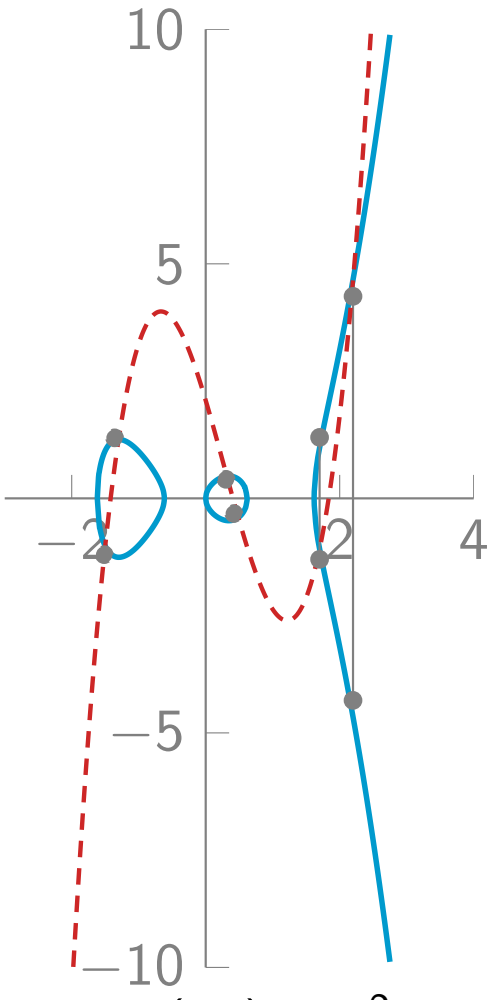
$E(\mathbb{F}_q) : y^2 = x^3 - 3x + 1$



$\#E(\mathbb{F}_q) \sim q$

Genus 2 addition

$\mathcal{C}_1(\mathbb{F}_q) : y^2 = x^5 - 3x^3 + x,$



$\#J_{\mathcal{C}}(\mathbb{F}_q) \sim q^2$

Scalar multiplication

- multiplication-by-m map: $P \mapsto [m]P$ on $E(\mathbb{F}_q)$, $\mathcal{D} \mapsto [m]\mathcal{D}$ on $J_{\mathcal{C}}(\mathbb{F}_q)$
- optimized binary double-and-add scalar multiplication:

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1. write m in binary rep. $m = \sum_{i=0}^{\log m - 1} m_i 2^i$, $m_i \in \{0, 1\}$
2. $R \leftarrow P$
3. for i from $\log m - 1$ to 0 do
   1. $R \leftarrow 2R$ (Doubling)
   2. if $m_i = 1$ then $R \leftarrow R + P$ (Addition)
4. return R

- cost: $\log m$ doublings $+ \sim \frac{1}{2} \log m$ additions in average

Multi-scalar multiplication

$$[m]P + [\ell]Q \in \mathbf{G} \subset E(\mathbb{F}_q)$$

1. write $m \leqslant \ell$ in binary rep. $m = \sum_{i=0}^{\log m - 1} m_i 2^i$, $\ell = \sum_{i=0}^{\log \ell - 1} \ell_i 2^i$, $m_i, \ell_i \in \{0, 1\}$
2. precompute $T = P + Q$
3. if $\log \ell > \log m$ then $R \leftarrow Q$
4. else $R \leftarrow T$
5. for i from $\log \ell - 1$ to 0 do
   1. $R \leftarrow 2R$ (Doubling)
   2. if $m_i = \ell_i = 1$ then $R \leftarrow R + T$ (Addition)
   3. else if $m_i = 1$ and $\ell_i = 0$ then $R \leftarrow R + P$ (Addition)
   4. else if $m_i = 0$ and $\ell_i = 1$ then $R \leftarrow R + Q$ (Addition)
6. return R

- cost: $\log \ell$ doublings $+ \sim \frac{3}{4} \log \ell$ additions in average

Algorithme GLV pour la multiplication scalaire

Assume there is an efficient (almost free) endomorphism

$$\phi : G \to G, \quad \phi(P) = \lambda_\phi P$$

λ_ϕ is large $\to$ decompose $m = m_0 + \lambda_\phi m_1 \bmod r$ with $\log m_0 \sim \log m_1 \sim \log m/2$

Multi-exponentiation

Compute $mP = m_0 P + m_1 \phi(P)$ in $(\log m)/2$ operations.

Save half doublings for a cost of a quarter of additions.

Endomorphisms: an example

$E_\alpha(\mathbb{F}_q) : y^2 = x^3 + \alpha x, \quad j(E_\alpha) = 1728$ (*i.e.* CM by $\sqrt{-1}$, $D = 4$)

- $q \equiv 1 \bmod 4$,
- let $i \in \mathbb{F}_q$ s.t. $i^2 = -1 \in \mathbb{F}_q$
- $\phi : (x, y) \mapsto (-x, iy)$ is an endomorphism
- $\phi \circ \phi(x, y) = (x, -y)$
- $\phi^2 + \mathrm{Id} = 0$ on $E(\mathbb{F}_q)$
- eigenvalue: $\lambda_\phi \equiv \sqrt{-1} \bmod \#E(\mathbb{F}_q)$
- this means for P of prime-order r, $\phi(P) = [\lambda_\phi \bmod r]P$

Endomorphism: Frobenius map

- Frobenius map, $E(\mathbb{F}_q)$, $(x, y) \in E(\mathbb{F}_{q^n}) \mapsto (x^q, y^q) \in E(\mathbb{F}_{q^n})$. Why ?
  - $E(\mathbb{F}_q) : y^2 = x^3 + a_4x + a_6,\ a_4, a_6 \in \mathbb{F}_q$
  - Not directly useful in this way. Used with twisted curves (Galbraith-Lin-Scott GLS curves)
- $j(E) = 1728, 8000, -3375 \longleftrightarrow \phi = \sqrt{-1}, \sqrt{-2}, \frac{1+\sqrt{-7}}{2}$.
- $j(E) = 0, 54000, -32768 \longleftrightarrow \phi = \frac{-1+\sqrt{-3}}{2}, \sqrt{-3}, \frac{1+\sqrt{-11}}{2}$.
- Galbraith-Lin-Scott (GLS) curves (2009): defined over $\mathbb{F}_{q^2}$ instead of $\mathbb{F}_q$, $j \in \mathbb{F}_q$, one endomorphism $\phi : \phi^2 = -\mathrm{Id}$ on $E(\mathbb{F}_{q^2})$.
  - but still $j \in \mathbb{F}_q$
- These are all available *fast* endomorphisms.

Implementation

Fast algorithms for scalar multiplication: GLV

Fast group law computation

Fast modular arithmetic : special primes (ex. $p = 2^{127} - 1$)

Example: No curve $E/\mathbb{F}_{q^2}$ with $p = 2^{127} - 1$ and GLV of dimension 4.

Challenge: the fastest implementation for a given security level

Our contribution

Four dimensional GLV via the Weil restriction

joint work with Aurore Guillevic

GLV friendly curve zoo

Genus 1

- GLV 2001 : complex multiplication by $\sqrt{-1}, \sqrt{-2}, \frac{1+\sqrt{-7}}{2}, \sqrt{-3}, \frac{1+\sqrt{-11}}{2}$.
- Galbraith-Lin-Scott 2009: curves/$\mathbb{F}_{q^2}$, $j \in \mathbb{F}_q$.
- Longa-Sica 2012: 4-dim GLV+GLS

Genus 2

- Mestre, Kohel-Smith, Takashima : explicit real multiplication by $\sqrt{2}$, $\sqrt{5}$
- 4-dim. : Buhler-Koblitz, Furukawa-Takahashi curves

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- Longa-Sica 2012: 4-dim GLV+GLS
- This work: 4 dim.-GLV on two families of curves/$\mathbb{F}_{q^2}$, but $j \in \mathbb{F}_{q^2}$.

Genus 2

- Mestre, Kohel-Smith, Takashima : explicit real multiplication by $\sqrt{2}$, $\sqrt{5}$
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4-GLV, . . ., 2^i-GLV: time-memory trade-off

- We would like a 4-dimensional decomposition of m when computing mP
- 2 endomophisms ϕ, ψ of eigenvalues $\lambda_\phi, \lambda_\psi$
- decompose $m \equiv m_1 + m_2\lambda_\phi + m_3\lambda_\psi + m_4\lambda_\phi\lambda_\psi \bmod r$ with $\log m_i \sim \frac{1}{4} \log m$
- Store $P, \phi(P), \psi(P), \phi\psi(P), \ldots \Rightarrow$ 16 points
- 4-dim. multiexponentiation $\rightarrow$ Save $\frac{3}{4} \log m$ doublings and $\sim \frac{17}{32} \log m$ additions.

Dimension 4 - Longa and Sica 2012

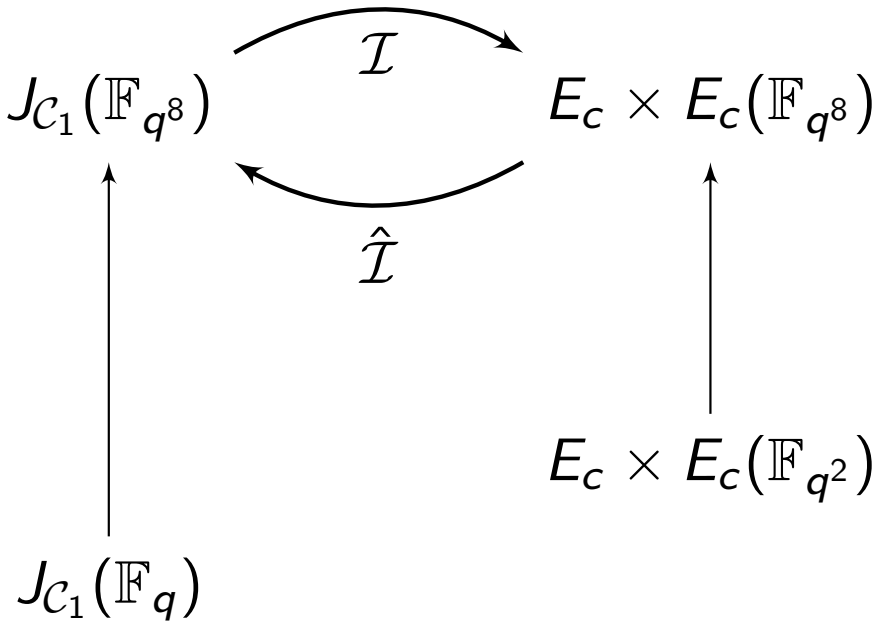
- Curves are ordinary, i.e. endomorphisms form a lattice of dimension 2 $\Rightarrow [1, \phi]$
- we need ψ s.t. $\lambda_\psi \equiv \alpha + \beta\lambda_\phi \bmod r$ and $\alpha, \beta > r^{1/4}$ to have a decomposition

How to construct ψ efficiently computable?

Longa-Sica curves (2012)

Consider GLS curves with small $D \to 2$ endomorphisms $\psi: \ \psi^2 + 1 = 0$, $\phi: \ \phi^2 + D = 0$ for points over $\mathbb{F}_{q^2}$.

Satoh's curves

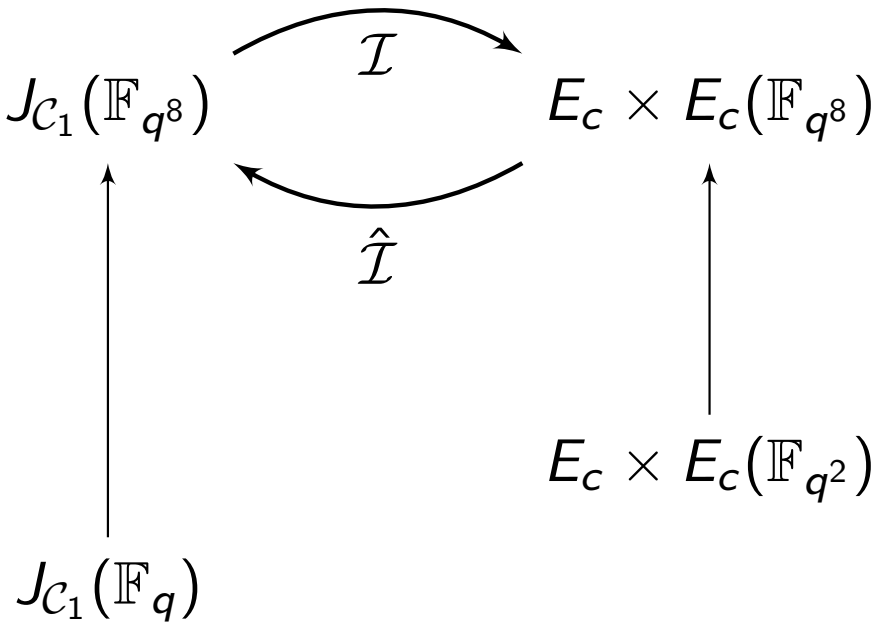


$$\mathcal{C}_1 \colon\ y^2 = x^5 + ax^3 + bx,\ a, b \in \mathbb{F}_q$$

$J_{\mathcal{C}_1}$ is the Weil restriction of

$$E_c/\mathbb{F}_{q^2} : y^2 = x^3 + 27(3c - 10)x + 108(14 - 9c), \quad c = a/\sqrt{b}$$

Satoh's Jacobians



$$D = 2D' \longrightarrow E_c \overset{\mathcal{I}_2}{\dashrightarrow} ?$$

We start by computing a degree 2 isogeny (i.e. a map between curves) $\mathcal{I}_2$ from E_c.

4-dim GLV on elliptic curves

We computed with Vélu's formulas this 2-isogeny

$$\begin{array}{rcl} \mathcal{I}_2 : E_c & \to & E_{-c} \\ (x,y) & \mapsto & \left(\frac{-x}{2} + \frac{162+81c}{-2(x-12)}, \frac{-y}{2\sqrt{-2}} \left(1 - \frac{162+81c}{(x-12)^2} \right) \right) \end{array}$$

- $E_c/\mathbb{F}_{q^2} : y^2 = x^3 + 27(3c - 10)x + 108(14 - 9c)$
- $E_{-c}/\mathbb{F}_{q^2} : y^2 = x^3 + 27(-3c - 10)x + 108(14 + 9c)$

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- In $\mathbb{F}_{q^2}$, $\pi_q(c) = -c$
- Go back from E_{-c} to E_c with the Frobenius map

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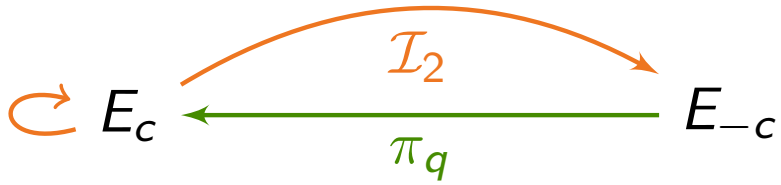
$\pi_q \circ \mathcal{I}_2$
$= \phi_2$
$\equiv [\sqrt{\pm 2}]$



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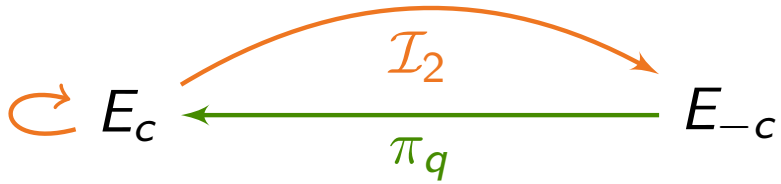

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- ϕ_2 is different from the CM

4-dim GLV on elliptic curves

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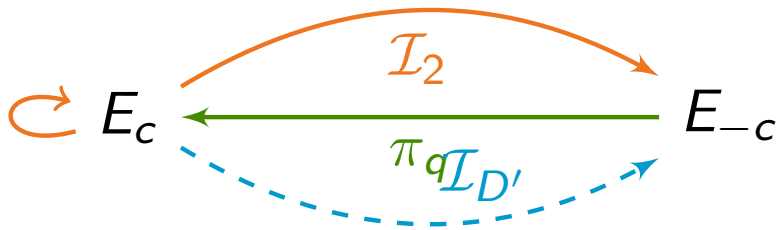

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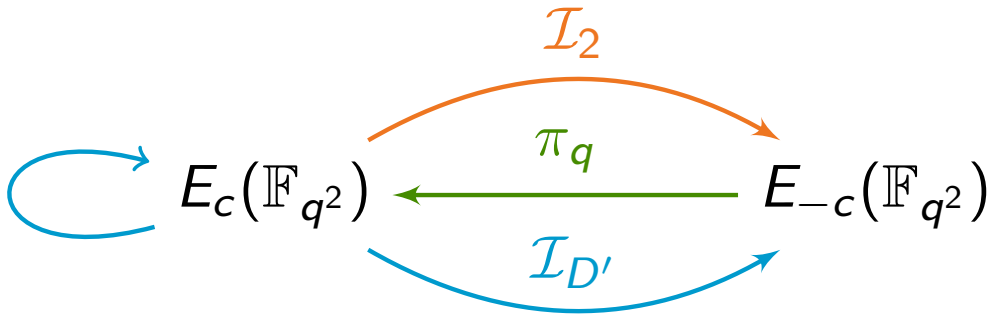
- In $\mathbb{F}_{q^2}$, $\pi_q(c) = -c$
- Go back from E_{-c} to E_c with the Frobenius map
- ϕ_2 is different from the CM
- We can construct a second endomorphism from CM.

Efficient 4-dim. GLV on E_c

$$\pi_q \circ \mathcal{I}_2 = \phi_2 \equiv [\sqrt{\pm 2}]$$
$$\pi_q \circ \mathcal{I}_{D'} = \phi_{D'} \equiv [\sqrt{\mp D'}]$$



- second isogeny $\mathcal{I}_{D'}$ computed with Velu's formulas
- 4-dimensional decomposition using proper values of $1, \phi_2, \phi_{D'}, \phi_2 \circ \phi_{D'}$.
- $\phi_2^2 \pm 2 = 0$, $\phi_{D'}^2 \mp D' = 0$ for points defined over $\mathbb{F}_{q^2}$.

Example with $D = 40$

- $D = 40 = 4 \cdot (2 \cdot 5)$
- $\#E_c(\mathbb{F}_{q^2})$ of the form $(-2n^2 - 20m^2 + 4)/4$, $4 \mid \#E_c(\mathbb{F}_{q^2})$
- search for m, n s.t. q is prime and $\#E_c(\mathbb{F}_{q^2})$ is almost prime.

$$
\begin{aligned}
n &= 0\text{x}55\text{d}23\text{edfa}6\text{a}1\text{f}7\text{e}4 \\
m &= 0\text{x}549906\text{b}3\text{eca}27851 \\
t &= -\ 0\text{xfaca}844\text{b}264\text{dfaa}353355300\text{f}9\text{ce}9\text{d}3\text{a} \\
q &= 0\text{x}9\text{a}2\text{a}8\text{c}914\text{e}2\text{d}05\text{c}3\text{f}2616\text{cade}9\text{b}911\text{ad} \\
r &= 0\text{x}1735\text{ce}0\text{c}4\text{fbac}46\text{c}2245\text{c}3\text{ce}9\text{d}8\text{da}0244\text{f}9059\text{ae}9\text{ae}4784\text{d}6\text{b}2\text{f}65\text{b}29\text{c}444309 \\
c^2 &= 0\text{x}40\text{b}634\text{aec}52905949\text{ea}0\text{fe}36099\text{cb}21\text{a}
\end{aligned}
$$

with q, r prime and $\#E_c(\mathbb{F}_{q^2}) = 4r$.

Operation count at the 128 bit security level

Curve	Method	Operation count	Global estim.
E_c	4-GLV, 16 pts.	$2748m+1668s$	$4416m$
$D = 4$ [LongaSica12]	4-GLV, 16 pts.	$1992m+2412s$	$4404m$
E_c	2-GLV, 4 pts.	$4704m+2976s$	$7680m$
$J_{\mathcal{C}_1}$	4-GLV, 16 pts.	$4500m+ 816s$	$5316m$
$J_{\mathcal{C}_1}$	2-GLV, 4 pts.	$7968m+1536s$	$9504m$
FKT [Bos et al. 13]	4-GLV, 16 pts.	$4500m+ 816s$	$5316m$
Kummer [Bos et al. 13]	–	$3328m+2048s$	$5376m$

Table : Benchmarks for scalar multiplication at 128 security level

Curve	Method	Timing in ms.
$E_{1,c}$ this work	4-GLV, 16 pts.	0.002202
E_1 Longa-Sica	4-GLV, 16 pts.	0.001882
$E_{1,c}$ GLV	2-GLV, 4pts.	0.004070
$J_{\mathcal{C}_1}$ this work	4-GLV, 4 pts.	0.001831