Spaces of sections on algebraic surfaces

Being (the other) half of a (relatively) recently defended thesis. . .

Hamish Ivey-Law

Supervisor: David Kohel   *Co-supervisor:* Claus Fieker

Institut de Mathématiques de Luminy
Université d'Aix-Marseille

School of Mathematics and Statistics
University of Sydney

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Spaces of sections on algebraic surfaces

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Introduction

- Given a divisor D on a curve C, the Riemann-Roch problem for D is the problem of calculating the dimension and determining a basis for the space of functions $L(C, nD)$ in terms of n.
- We will consider the analogous problem on certain classes of surfaces: Given a formal linear combination $mD_1 + nD_2$ of curves on a surface X, we calculate the dimension and determine a basis of the space of functions $H^0(X, mD_1 + nD_2)$ in terms of m and n.
- We consider the two cases: $X = C \times C$ and $X = \mathrm{Sym}^2(C)$ where C is a hyperelliptic curve of genus $g \geqslant 2$.

Definitions: Square of the curve

- k a field of characteristic not 2.
- C a hyperelliptic curve of genus $g \geqslant 2$.
- $C^2 = C \times C$ the square of C.
- Fix a Weierstrass point $\infty \in C(\overline{k})$
- $V_\infty = \{\infty\} \times C$ the vertical embedding of C in C^2.
- $H_\infty = C \times \{\infty\}$ the horizontal embedding of C in C^2.
- $F = 2(V_\infty + H_\infty)$.
- Δ and ∇ the diagonal and antidiagonal embeddings of C in C^2.
- $D_\infty = 2(\infty)$ or $D_\infty = (\infty^+) + (\infty^-)$ depending on whether C has one or two points at infinity.
- Let D_∇ be the image of D_∞ on ∇.

Definitions: Symmetric square of the curve

- $S = C^2/\langle\sigma\rangle$ the symmetric square of C and

$$\pi : C^2 \to S$$

  is the quotient map.
- $\Delta_S = \pi(\Delta)$,
- $\nabla_S = \pi(\nabla)$ and
- $\Theta_S = \pi(V_\infty) = \pi(H_\infty)$ are the (scheme-theoretic) images under the quotient map.
- Note that $2\Theta_S$ is a k-rational divisor even though Θ_S is not k-rational in general.

The Néron-Severi group

- Recall that the *Picard group* of a variety X, denoted by $\mathrm{Pic}(X)$, is the group of divisors of X modulo rational (linear) equivalence, and $\mathrm{Pic}^0(X)$ is the subgroup of divisors algebraically equivalent to zero.
- The *Néron-Severi group* is

$$\mathrm{NS}(X) = \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X);$$

  equivalently it is the group of divisors of X modulo algebraic equivalence.
- Néron-Severi Theorem: The Néron-Severi group is a finitely generated abelian group.
- Matsusaka's Theorem: The torsion subgroup of the Néron-Severi group is finite.

The Néron-Severi group of C^2

- If C is a curve, then $\mathrm{NS}(C) \cong \mathbb{Z}$ (isomorphism given by the degree map).
- For any two curves C_1 and C_2, we have

$$\mathrm{NS}(C_1 \times C_2) \cong \mathrm{NS}(C_1) \times \mathrm{NS}(C_2) \times \mathrm{Hom}(J_{C_1}, J_{C_2}).$$

- So $\mathrm{NS}(C_1 \times C_2) \cong \mathbb{Z}^{2+\rho}$ where $1 \leqslant \rho \leqslant 4g_1g_2$.

The Néron-Severi group of S

Proposition

With S as above,

$$\mathrm{NS}(S) \cong \mathbb{Z}^{1+\rho} \times (\mathbb{Z}/2\mathbb{Z})^{\tau}$$

where $1 \leqslant \rho \leqslant 4g^2$ and $0 \leqslant \tau < \infty$.

- Questions I didn't get around to answering:
  - When is $\tau > 0$? How big can it be?
  - What is in $\mathrm{NS}(S)_{\mathrm{tors}}$? (Wild guess: Maybe divisors *corresponding* to non-scalar, self-dual endomorphisms of J_C?)

Subgroups of NS(C^2) and NS(S)

Let m and r be non-negative integers.

- (The classes of) V_∞, H_∞ and ∇ are linearly independent in NS(C^2).
- We will consider the divisors of the form $mF + r\nabla$ in Div(C^2) (where $F = 2(V_\infty + H_\infty)$).
- Divisors of this form don't span NS(C^2).
- There is a relation
$$F \sim \Delta + \nabla$$
coming from the function $x_1 - x_2$ on C^2 where $k(C^2) = k(x_1, y_1, x_2, y_2)$.

Subgroups of Div(C^2) and Div(S)

Let m and r be non-negative integers.

- (The classes of) Θ_S and ∇_S are linearly independent in NS(S).
- We will consider divisors of the form $2m\Theta_S + r\nabla_S$ in Div(S).
- Divisors of this form don't span NS(S).
- There is a relation

$$4\Theta_S \sim \Delta_S + 2\nabla_S$$

coming from the function $(x_1 - x_2)^2$ on S.

Fundamental exact sequence

Throughout we fix $\gamma = g - 1$.

Let m and r be non-negative integers. Then

$$\begin{aligned} 0 \to \mathscr{O}_{C^2}(mF + (r-1)\nabla) \\ \to \mathscr{O}_{C^2}(mF + r\nabla) \\ \to \mathscr{O}_{\nabla}((2m - \gamma r)D_{\nabla}) \to 0 \end{aligned}$$

is an exact sequence (because $\mathscr{O}_{C^2}(mF + r\nabla) \otimes \mathscr{O}_{\nabla} \cong \mathscr{O}_{\nabla}((2m - \gamma r)D_{\nabla})$).

Fundamental exact sequence

We thus obtain a long exact sequence of cohomology

$$\begin{aligned}
0 &\to H^0(C^2, mF + (r-1)\nabla) \\
&\to H^0(C^2, mF + r\nabla) \\
&\to H^0(\nabla, (2m - \gamma r)D_\nabla) \\
&\to H^1(C^2, mF + (r-1)\nabla) \\
&\to H^1(C^2, mF + r\nabla) \\
&\to H^1(\nabla, (2m - \gamma r)D_\nabla) \to \cdots
\end{aligned}$$

The easy cases (i): $2m - \gamma r > 0$

- If $2m - \gamma r > 0$, then we can show that $H^1(C^2, mF + (r-1)\nabla) = 0$ by showing that it is surrounded by zeros in the long exact sequence of cohomolgy:
  - $r = 1$: Apply the Künneth formula to obtain
    $$H^1(C^2, mF) \cong (H^0_C \otimes H^1_C) \oplus (H^1_C \otimes H^0_C)$$
    where H^i_C denotes $H^i(C, mD_\infty)$. Then $H^1_C = 0$ by Serre duality (since $m > \gamma$).
  - $r \geqslant 2$: Assume for induction that $H^1(C^2, mF + (r-2)\nabla) = 0$. Then from the long exact sequence of cohomology, it suffices to prove that
    $$H^1(\nabla, (2m - \gamma(r-1))D_\nabla) = 0.$$
    But this follows from Serre duality since $K_\nabla - (2m - \gamma(r-1))D_\nabla = -(2m - \gamma r)D_\nabla$.
- Thus the sequence splits:
  $$H^0(C^2, mF + r\nabla) \cong H^0(C^2, mF + (r-1)\nabla) \oplus H^0(\nabla, (2m - \gamma r)D_\nabla).$$

The easy cases (ii): $2m - \gamma r < 0$

- Since D_∇ is effective, $H^0(\nabla, (2m-\gamma r)D_\nabla) = 0$ if $2m - \gamma r < 0$. Thus in this case
$$H^0(C^2, mF + (r-1)\nabla) \cong H^0(C^2, mF + r\nabla).$$
- It remains to consider the case $2m - \gamma r = 0$.

The split long exact sequence

Suppose $2m - \gamma r = 0$.

- $H^0(\nabla, (2m - \gamma r)D_\nabla) = H^0(\nabla, \mathscr{O}_\nabla)$ has dimension 1.
- $H^1(C^2, mF + (r-1)\nabla)$ is not necessarily zero.
- In the next section, we will describe an algorithm that calculates an explicit basis for $H^0(C^2, mF + r\nabla)$ for any *particular* curve, thus allowing us to verify exactness in any particular case.
- Unfortunately I was unable to prove exactness in this case in general.
- Testing using the aforementioned algorithm has not turned up a counterexample after many (1000s of) tries. Hence...

Conjecture

When $2m - \gamma r = 0$, the map $H^0(\nabla, \mathscr{O}_\nabla) \to H^1(C^2, mF + (r-1)\nabla)$ is zero and so we obtain an exact sequence

$$0 \to H^0(C^2, mF + (r-1)\nabla) \to H^0(C^2, mF + r\nabla) \to H^0(\nabla, \mathscr{O}_\nabla) \to 0.$$

To simplify the exposition, we assume henceforth that the conjecture holds.

Structure of $H^0(C^2, mF + r\nabla)$

Theorem (I.-L.)

Let m and r be integers satisfying $m > \gamma$ and $r \geqslant 0$. We have

$$H^0(C^2, mF + r\nabla) \cong H^0(C^2, mF) \oplus \bigoplus_{i=1}^{r} H^0(\nabla, (2m - \gamma i)D_\nabla).$$

Corollary (I.-L.)

$$h^0(C^2, mF + r\nabla) = \begin{cases} (2m-\gamma)^2 + 4mr - \gamma r(r+2) & \text{if } \gamma < 2m - \gamma r, \\ (2m-\gamma)^2 + 4mr - \gamma r(r+1) - 2m + g & \text{if } 0 < 2m - \gamma r \leqslant \gamma, \\ (2m-\gamma)^2 + 2m(r-2) + g + 1 & \text{if } 2m - \gamma r = 0, \text{ and} \\ h^0(C^2, mF + \left\lfloor \frac{2m}{\gamma} \right\rfloor \nabla) & \text{if } 2m - \gamma r < 0. \end{cases}$$

Intermezzo: Intersection pairing and Euler characteristic

Recall that Riemann-Roch for surfaces says that for any divisor D on a surface X we have

$$\begin{aligned}\chi(D) &= h^0(X, D) - h^1(X, D) + h^2(X, D) \\ &= \frac{1}{2} D \cdot (D - K_X) + \chi(\mathscr{O}_X)\end{aligned}$$

where K_X is a canonical divisor on X and

$$\cdot : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

is the *intersection pairing* on X.

Intersection pairing and Euler characteristic

Proposition

The intersection pairing on $\mathrm{Div}(C^2) \times \mathrm{Div}(C^2)$ *is given by the following table:*

$\cdot$	V_∞	H_∞	Δ	∇
V_∞	0	1	1	1
H_∞	1	0	1	1
Δ	1	1	$2-2g$	$2+2g$
∇	1	1	$2+2g$	$2-2g$

Let $D = mV_\infty + nH_\infty + r\nabla$ *be a divisor on* C^2. *Then*

$$\chi(D) = (m-\gamma)(n-\gamma) + r(m+n) - \gamma r(r+2).$$

The higher cohomology groups...

Corollary

Let $m > \gamma$ and $r \geqslant 0$ be integers. Then

$$h^1(C^2, mF + r\nabla) = \begin{cases} 0 & \text{if } \gamma < 2m - \gamma r, \\ g - (2m - \gamma r) & \text{if } 0 < 2m - \gamma r \leqslant \gamma, \\ g + 1 & \text{if } 2m - \gamma r = 0, \text{ and} \\ h^1(C^2, mF + \left\lfloor \frac{2m}{\gamma} \right\rfloor \nabla) & \text{if } 2m - \gamma r < 0. \end{cases}$$

and

$$h^2(C^2, mF + r\nabla) = 0$$

Structure of $H^0(S, 2m\Theta_S + r\nabla_S)$

Theorem (I.-L.)

Let m be an integer with $m > \gamma$. Then for all integers $r \geqslant 0$,

$$H^0(S, 2m\Theta_S + r\nabla_S) \cong H^0(S, 2m\Theta_S) \oplus \bigoplus_{i=1}^{r} H^0(\mathbb{P}^1, (2m - \gamma i)(\infty)).$$

Corollary (I.-L.)

If $2m - \gamma r \geqslant 0$, then

$$h^0(S, 2m\Theta_S + r\nabla_S) = \frac{(2m-\gamma)(2m-\gamma+1)}{2} + r(2m+1) - \gamma\frac{r(r+1)}{2}.$$

Otherwise

$$h^0(S, 2m\Theta_S + r\nabla) = h^0(S, 2m\Theta_S + \left\lfloor \frac{2m}{\gamma} \right\rfloor \nabla).$$

Intersection pairing and Euler characteristic

Proposition

The intersection pairing on $\mathrm{Div}(S) \times \mathrm{Div}(S)$ *is given by the following table:*

$\cdot$	Θ_S	Δ_S	∇_S
Θ_S	1	2	1
Δ_S	2	4 - 4g	2 + 2g
∇_S	1	2 + 2g	1 - g

If $D = m\Theta_S + r\nabla_S$ *is an element of* $\mathrm{Div}(S)$, *then*

$$\chi(D) = \frac{(m-\gamma)(m-\gamma+1)}{2} + r(m+1) - \gamma\frac{r(r+1)}{2}.$$

The higher cohomology groups...

Corollary

Let $m > \gamma$ and $r \geqslant 0$ be integers. Then

$$h^1(S, 2m\Theta_S + r\nabla_S) = (r - r')\left(\frac{\gamma}{2}(r + r' + 1) - (2m + 1)\right)$$

where $r' = \min\{r, \left\lfloor \frac{2m}{\gamma} \right\rfloor\}$. In particular, $h^1(S, 2m\Theta_S + r\nabla_S) = 0$ if $0 \leqslant 2m - \gamma r$. Furthermore,

$$h^2(S, 2m\Theta_S + r\nabla_S) = 0.$$

Eigenspace decomposition

Goal: an explicit basis for $H^0(S, 2m\Theta_S + r\nabla_S)$.

Proposition

For any divisor D on $S = C^2/\langle\sigma\rangle$,

$$H^0(S, D) \cong H^0(C^2, \pi^* D)^{\langle\sigma\rangle}.$$

Since $\pi^*(2m\Theta_S + r\nabla_S) = mF + r\nabla$, we reduce to the problem of computing $H^0(C^2, mF + r\nabla)^{\langle\sigma\rangle}$.

Eigenspace decomposition

Lemma

Let $W^{\varepsilon}_{m,r}$ denote the subspace of $H^0(C^2, mF - r\Delta)$ on which σ acts by $\varepsilon = \pm 1$. Then

$$H^0(C^2, mF + r\nabla)^{\langle\sigma\rangle} \cong W^{(-1)^r}_{m+r,r}.$$

This follows from the isomorphism

$$H^0(C^2, mF + r\nabla) \cong H^0(C^2, (m+r)F - r\Delta)$$

obtained from the relation $F \sim \Delta + \nabla$.

- We have reduced the problem to finding a basis of $W^{(-1)^r}_{m+r,r}$.
- We can show that

$$W^{+1}_{m+r,r} = H^0(C^2, (m+r)F - r\Delta)^{\langle\sigma\rangle}$$
$$W^{-1}_{m+r,r} = (x_1 - x_2)H^0(C^2, (m+r-1)F - r\Delta)^{\langle\sigma\rangle}$$

are subspaces of $H^0(C^2, (m+r)F) \cong H^0(C, (m+r)D_\infty)^{\otimes 2}$ of sections with valuation at least r on Δ.

Hasse derivatives

- Let A be a ring and let $j \geqslant 0$ be an integer. The *jth Hasse derivative* of a polynomial $w = \sum_{i=0}^{n} a_i t^i$ in $A[t]$ is defined to be

$$D_t^{(j)} w = \sum_{i=j}^{n} \binom{i}{j} a_i t^{i-j}.$$

- When $\mathrm{char}(A)$ is coprime to $j!$ we have $D_t^{(j)} w = \frac{1}{j!} \frac{d^j}{dt^j} w$, where $\frac{d}{dt} w$ is the usual formal derivative of a polynomial. In particular, $D_t^{(0)} w = w$ and $D_t^{(1)} w = \frac{d}{dt} w$ for all w in $A[t]$, however $D_t^{(i)} D_t^{(j)} w \neq D_t^{(i+j)} w$ in general.
- Let A be a ring, let a be in A, and let w be an element of $A[t]$. Then

$$w = \sum_{i=0}^{\deg(w)} (D_t^{(i)} w)(a)(t-a)^i.$$

Hasse derivatives

Proposition (I.-L.)

As before let C be the curve $y^2 - f(x)$, let $k(C^2) = k(x_1, x_2, y_1, y_2)$ be the function field of C^2 and set $t = \frac{1}{2}(x_1 - x_2) \in k(C^2)$. For $i = 1, 2$ and for all $j > 0$ we have

$$\begin{aligned} D_t^{(j)} x_1^m &= \binom{m}{j} x_1^{m-j} \\ D_t^{(j)} x_2^m &= (-1)^j \binom{m}{j} x_2^{m-j} \\ D_t^{(j)} y_i &= \frac{1}{2f(x_i)} \left(D_t^{(j)} f(x_i) - \sum_{\ell=1}^{j-1} D_t^{(\ell)} y_i D_t^{(j-\ell)} y_i \right) y_i \\ &= \frac{G_i^{(j)}(x_i)}{(2f(x_i))^j} y_i \end{aligned}$$

where $G_i^{(j)}$ is a polynomial in $k[x_i]$ of degree at most $j(\deg(f) - 1)$.

In a neighbourhood of Δ

- Any section $w \in H^0(C^2, mF)$ has the form
$$w = a + by_1 + cy_2 + dy_1y_2$$
where a, b, c, d are polynomials in $k[x_1, x_2]$ (of degree bounded by m).
- For any $w \in H^0(C^2, (m+r)F)$ we can consider the formal expansion
$$w = \sum_{i=0}^{\infty} D_t^{(i)} w\Big|_{\Delta} t^i$$
in a neighbourhood of Δ. Here
  - $t = \frac{1}{2}(x_1 - x_2)$ generates the maximal ideal $\mathfrak{m}_\Delta$ in the local ring $\mathscr{O}_{C^2,\Delta}$, and
  - $D_t^{(i)} w\Big|_{\Delta}$ denotes the image of $D_t^{(i)} w$ under the quotient
$$\mathscr{O}_{C^2,\Delta} \to \mathscr{O}_{C^2,\Delta}/\mathfrak{m}_\Delta \cong k(\Delta).$$
- A section with valuation at least r on Δ is one for which $D^{(i)} w\Big|_{\Delta} = 0$ for $i = 0, \ldots, r-1$.

An explicit description of the basis

- Define
$$\varphi_i : W^{(-1)^r}_{m+r,0} \to k(\Delta)$$
by sending a section $w \in W^{(-1)^r}_{m+r,0} \subset H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(s))$ to $D^{(i)}w\Big|_{\Delta}$ (here s is of order m).
- The image lies in a finitely generated subring.
- φ_i is linear (being just a derivative and evaluation) and (after fixing bases) is given by a vector in k^u for some u (of order m^2).

Proposition (I.-L.)

$$W^{(-1)^r}_{m+r,r} = \bigcap_{i=0}^{r-1} \mathrm{Ker}(\varphi_i).$$

Applications

- Having an explicit basis allows us to verify the conjecture of the previous section in any particular case.
- If C has genus $g = 2$, we obtain (a projective linear transformation of) the well-known embedding of J_C in $\mathbb{P}^{15}$ published by Cassels and Flynn. In the present work, this corresponds to calculating a basis of the space $H^0(S, 4\Theta_S + 4\nabla_S)$.
- The Fujita conjecture (proved for surfaces by Reider) says:
  - Let X be a smooth projective variety of dimension n, let K_X be a canonical divisor on X and let H be an ample divisor on X. Then $K_X + \lambda H$ is very ample if and only if $\lambda \geqslant n + 2$.
  - We can show that $K_{C^2} = \gamma F$ is a canonical divisor on C^2 and $K_S = 2(g-2)\Theta_S + \nabla_S$ is a canonical divisor on S.
  - Hence we can now explicitly give several new embeddings of C^2 and S.
- Codes on C^2 and S:
  - Bases of $H^0(C^2, mF + r\nabla)$ and $H^0(S, 2m\Theta_S + r\nabla_S)$ can be used to define codes.
  - This opens the door to studying codes on these surfaces.

Avenues for generalisation

There are several possible generalisations we might pursue:

- Similar results for elliptic curves are probably trivial to determine.
- Similar results for non-hyperelliptic curves are probably easy to determine: Difference is that ∇ is more complicated.
- Given a relatively explicit description of $\mathrm{End}(J_C)$ in terms of the intersection theory of the correspondences, can we find dimension formulae and explicit bases for arbitrary divisors on these surfaces? At least the Frobenius divisor in positive characteristic?
- Characteristic 2 will require new techniques.
- Higher symmetric products would allow us produce the birational maps $C^{(g)} \to J_C$ to the Jacobian, but requires a much more sophisticated theory.

Merci pour votre attention!

Thank you for your attention.