Discrete Logarithms in Medium Characteristic Finite Fields

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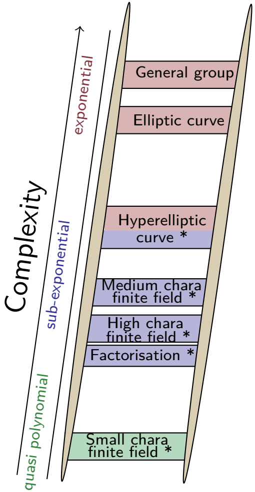

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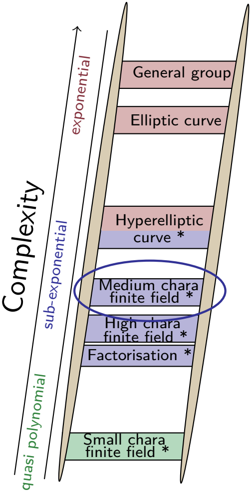

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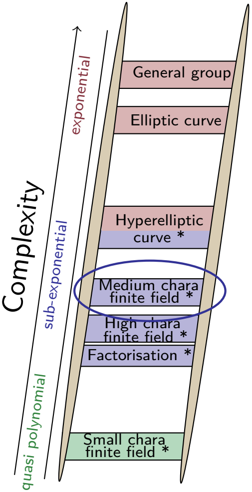

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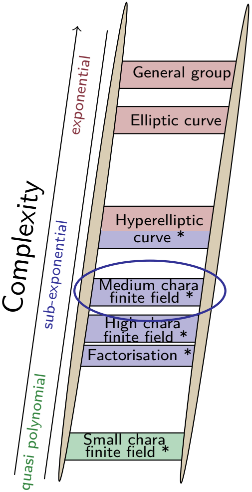

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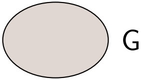

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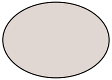
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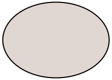

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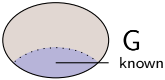

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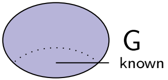

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$$\prod_{b_i \in B_2} v_1(b_i) = \prod_{b_i \in B_2} v_2(b_i) \quad \text{thanks to morphisms.}$$

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Factor base ? B_i:= prime ideals (of the ring of integers) with a norm smaller than a certain smoothness* bound.

*An ideal $\mathfrak{I}$ is B-smooth if all its factors have norms lower than B.

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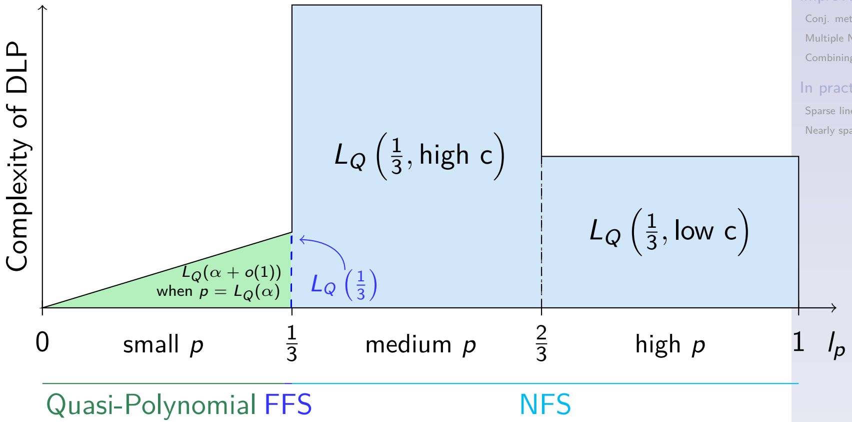

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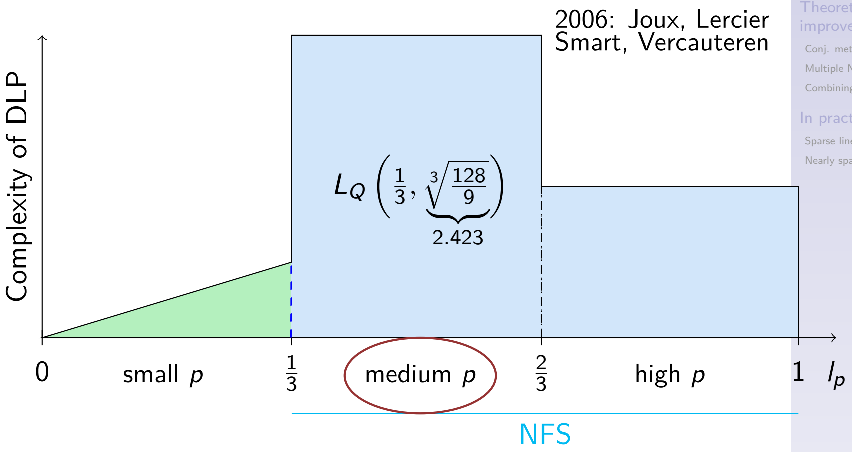

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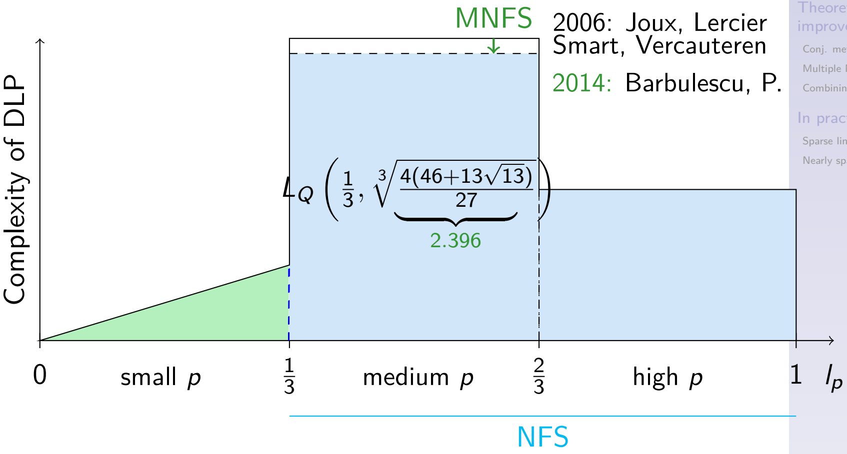

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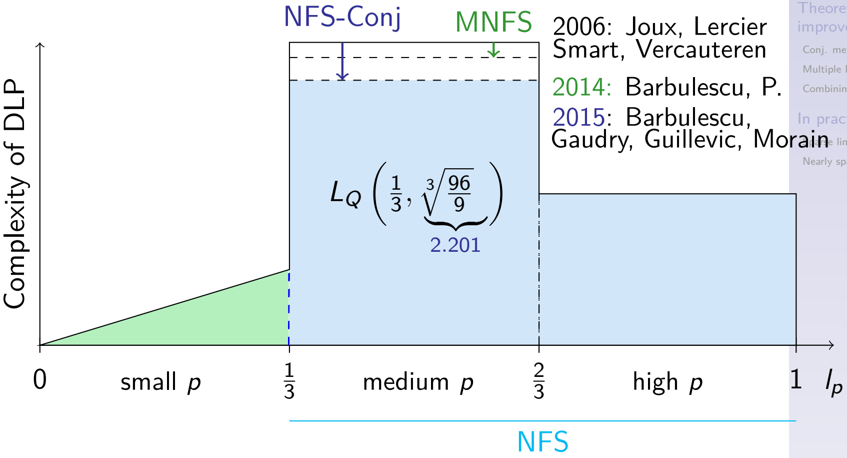

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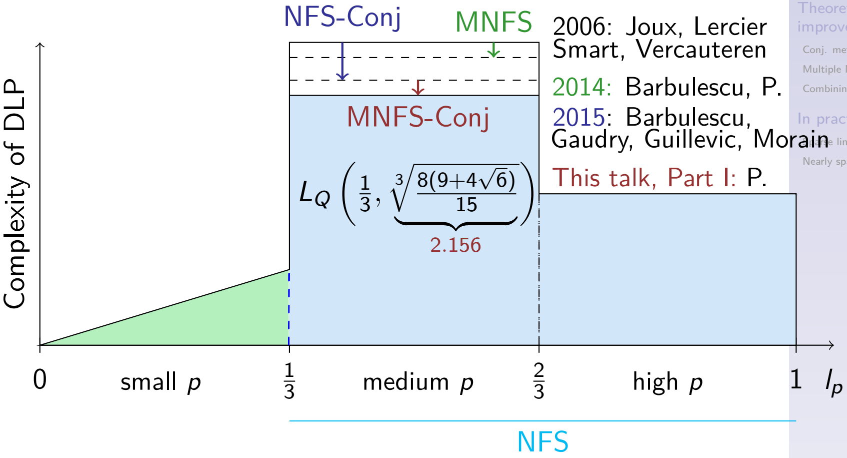
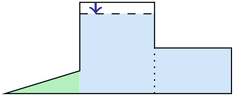


Part I, Asymptotic Complexity downturn: MNFS-Conj

Polynomial Selection

NFS-Conj



Preliminaries to the diagram:

Find two polynomials f_1 and f_2 with
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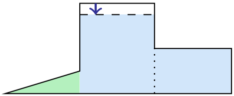

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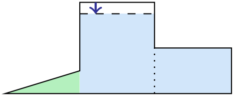

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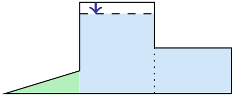

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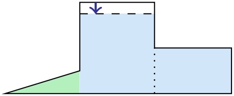

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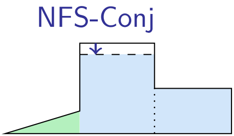

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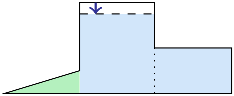

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New polynomial selection proposed by Barbulescu, Gaudry, Guillevic and Morain: the Conjugation Method.

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Degrees

The Conjugation Method

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small

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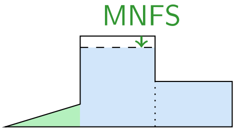
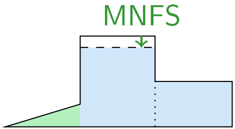
Degrees

Coeffs.



The Multiple Number Field Sieve

- Idea from integer factorization [Coppersmith 93], prime fields [Matyukhin 03], high and medium characteristic [Barbulescu, P. 14].



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$$\begin{tikzcd}
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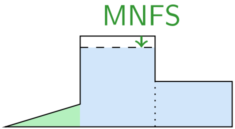

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- Choice of poly. f_1 and f_2 with a common root m in $\mathbb{F}_{p^n}$
  $\Rightarrow$ linear combination of f_1 and f_2
  $\Rightarrow$ for $i = 3, \dots, V$: $f_i = \alpha_i f_1 + \beta_i f_2$ with $\alpha_i, \beta_i \approx \sqrt{V}$.

Dissymetric MNFS in one slide

Dissymmetric = when a polynomial is better than the other.

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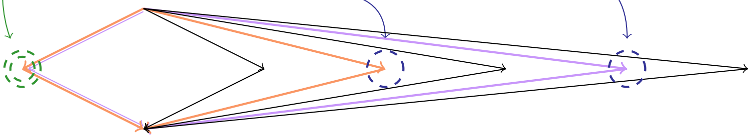

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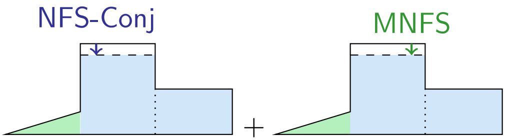

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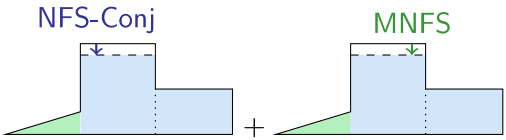
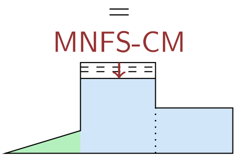


$\Rightarrow$ Best algorithm to solve the DLP in medium characteristic finite fields $\mathbb{F}_{p^n}$.



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  $\}$ How to catch it ?

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n $\sqrt{p}$

n $\sqrt{p}$

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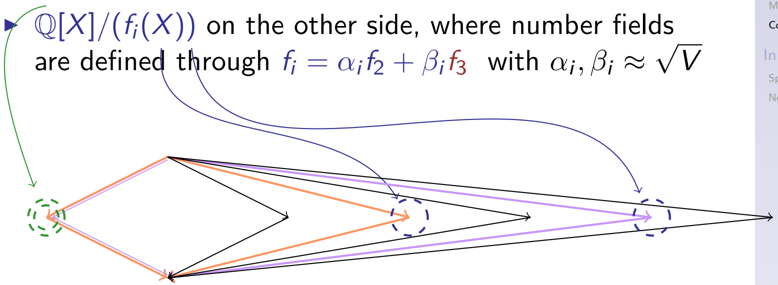
And then ?

Construct a Multiple NFS thanks to:

- $\mathbb{Q}[X]/(f_1(X))$ on one side
- $\mathbb{Q}[X]/(f_i(X))$ on the other side, where number fields are defined through $f_i = \alpha_i f_2 + \beta_i f_3$ with $\alpha_i, \beta_i \approx \sqrt{V}$



Asymptotic Complexity Analysis

The idea is classical:

1. Choose parameters of size:
   - Sieving space : $L_Q(1/3)$
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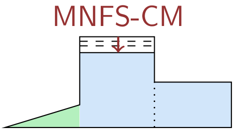
$$\Rightarrow L_Q\left(\frac{1}{3}, \sqrt[3]{\frac{8(9+4\sqrt{6})}{15}}\right)$$



Concrete impact

Complexity $\searrow$ from $L_Q(1/3, 2.201)$ to $L_Q(1/3, 2.156)$.
Is it a lot?

- $\ell \leftarrow$ security level we need
  $Q \leftarrow$ order of the associated target finite field.
  With previous algorithms: $\ell = L_Q(1/3, 2.201)$.
- Now, To get Q' such that $\ell = L_{Q'}(1/3, 2.156)$ we need:
  - $(2.156)^3 \log Q' (\log\log Q')^2 = (2.201)^3 \log Q (\log\log Q)^2$
  - so $\log Q' (\log\log Q')^2 \approx 1.064 \log Q (\log\log Q)^2$
  - it yields $\log Q' \approx 1.064 \log Q$.

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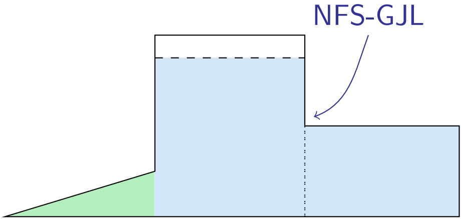
  - so $\log Q'(\log\log Q')^2 \approx 1.064 \log Q(\log\log Q)^2$
  - it yields $\log Q' \approx 1.064 \log Q$.

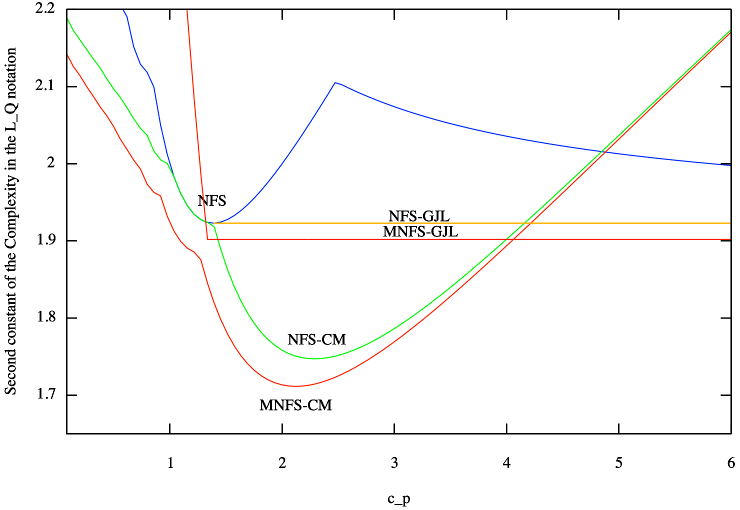
$\Rightarrow$ Increase the bitsize of the finite field by 6.4% to get the same security level.

- the Generalized Joux-Lercier Method [BGGM 15]
- with MNFS.



$$p = L_{p^n}(2/3, c_p)$$

Complexities at $p = L_{p^n}(2/3, c_p)$

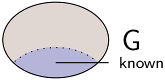


Part II, Practical improvement: Nearly Sparse Linear Algebra.

A joint work with Antoine Joux.

Index Calculus Algorithms

If you want to compute Discrete Logs in G:



1. Collection of Relations (or Sieving Phase)
   $\rightarrow$ Create a lot of sparse multiplicative relations between some (small) specific elements = the factor base

$$\prod g_i^{e_i} = \prod g_i^{e'_i} \quad \Rightarrow \quad \sum (e_i - e'_i) \log(g_i) = 0$$

   $\rightarrow$ So a lot of sparse linear equations
2. Linear Algebra
   $\rightarrow$ Recover the Discrete Logs of the factor base
3. Individual Logarithm Phase
   $\rightarrow$ Recover the Discrete Log of an arbitrary element

Linear Algebra and Index Calculus

- Matrix over finite sets.
- Sparse matrices = the major part of the entries = 0. Often: nbr of non zero coeffs per row is bounded by a constant, let us say K.

Some famous examples

- Factoring. Seek for a non trivial elt of the kernel of a matrix mod 2.
- Discrete log. Last records in small charac. for instance.

Advantages ?

- Less memory
- Specific algorithms

Sparse Linear Algebra

Sparse Linear Algebra

How to use less memory: for any non zero coeff. in a row, let memorize its column number and its value together.

Example

With $\mathbb{F}_7$ and $K = 3$.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Sparse Linear Algebra

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Sparse Linear Algebra, Naive method

We want to solve $Mx = 0$.
Let us manage a simple Gaussian Elimination.

$$\left(\begin{array}{cccccccc|c}
1 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\
2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 0 & 0 & 2 & 0 & 0
\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}
[1,1] & [5,3] & [8,2] & 0 \\
[1,2] & [3,1] & [6,2] & 0 \\
[4,4] & [7,1] & [0,0] & 0 \\
[4,3] & [8,1] & [0,0] & 0 \\
[1,1] & [2,1] & [0,0] & 0 \\
[3,5] & [8,2] & [0,0] & 0 \\
[2,5] & [6,6] & [0,0] & 0 \\
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it overflows the available memory!
$\rightarrow$ Stupid method.

Sparse Linear Algebra, specific algorithms

- Adapted Gaussian Elimination
  = choose pivots that minimize the loss of sparsity

$$
\left(\begin{array}{cccccccc|c}
1 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\
2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
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\end{array}\right)
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[1,1] & [5,3] & [8,2] & 0 \\
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\end{array}\right)
$$

- or, without any modification of the matrix, using matrix-by-vector multiplications only:
  - Krylov Subspace methods
  - Wiedemann algorithm(s)

Wiedemann

1986

Problem

Solve:

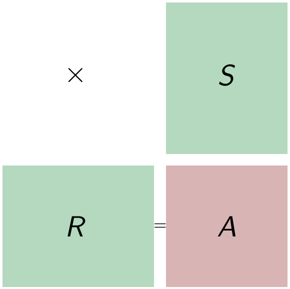
$$Sx = 0 \quad or \quad Sx = y$$

with S a sparse matrix with coefficients in a ring $\mathbb{K}$,
K non zero coeffs. per row max,
N= max(# rows, # col)

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \begin{matrix} K = 3 \\ N = 9 \end{matrix}$$

Wiedemann

1. Preconditioning step : We transform S into a square matrix A.



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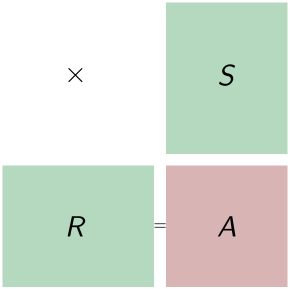

Why?

- Powers of A are well defined.
- A not sparse but multiplying $R\ S = A$ with a vector is quick: $O(KN)$
- $S.x = 0 \Rightarrow A.x = 0$ (or $S.x = y \Rightarrow A.x = y' = R.y$). The converse is true for almost all random matrices R.

Try to solve $A.x = 0$ (or $A.x = y'$).

Wiedemann

2. Computation of a scalar sequence : $({}^t w A^i v)_{i=0,\cdots,2n}$ with v, w two random vectors and $n = \#$ col. of A.
3. Reconstruction of the minimal polynomial of A.

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Why does 2 help 3 ?

- Cayley-Hamilton theorem: the characteristic polynomial of A, of degree n, annihilates A.
- so we seek for a_i s.t. $\sum_{i=0}^{n} a_i A^i = 0$. $(\star_1)$

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We have found a_i s.t. $\sum_{i=0}^{n} a_i A^i = 0$.

Wiedemann

4. Computation of the solution.
   - How to solve $Ax = 0$ thanks to $\sum_{i=0}^{n} a_i A^i = 0$?
     If there is a solution then $a_0 = 0$.
     So for a random vector r:
     $$\sum_{i=1}^{n} a_i A^i r = 0 \Leftrightarrow A \underbrace{\left(\sum_{i=1}^{n} a_i A^{i-1} r \right)}_{\text{Here is } x \text{ !}} = 0$$

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- How to solve $Ax = y$ thanks to $\sum_{i=0}^{n} a_i A^i = 0$?
  A inversible permits to assume $a_0 \neq 0$.
  So $\displaystyle\sum_{i=0}^{n} a_i A^i x = 0 \Leftrightarrow -a_0 x = \sum_{i=1}^{n} a_i A^i x$
  $\Leftrightarrow x = -(1/a_0) \sum_{i=1}^{n} a_i A^{i-1} A x$
  $\Leftrightarrow x = -(1/a_0) \sum_{i=1}^{n} a_i A^{i-1} y$. Here is x again !

Wiedemann

1. Preconditioning step: Transformation of $\boxed{S}$ into $\boxed{A}$.
   The problem becomes:
   $$A.x = 0 \quad \text{or} \quad A.x = y'.$$
2. Computation of a scalar sequence: $({}^t wA^i v)_{i=0,\cdots,2n}$
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   Complexity: quasi-linear in N (with fast B-M. algo).
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Final asymptotic complexity:

$$O(KN^2)$$

Let us parallelize!

- 1994. Coppersmith. Distributed computations for sparse linear algebra over $\mathbb{F}_2$.

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- 2002. Thomé. Generalized fast Berlekamp-Massey.

From Wiedemann to Block Widemann

1. Preconditioning step: Transformation of $\boxed{S}$ into a square matrix $\boxed{A}$. The problem becomes:

$$A.x = 0 \quad \text{ou} \quad A.x = y'.$$

2. Computation of a scalar sequence: $({}^t w A^i v)_{i=0,\cdots,2n}$ with v, w two random vectors

3. Reconstruction of the minimal polynomial of $\boxed{A}$ thanks to Berlekamp-Massey algorithm.

4. Computation of the solution.

From Wiedemann to Block Widemann

1. Preconditioning step: Transformation of S into a square matrix A. The problem becomes:

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2. Computation of a matrix sequence: $({}^tW\ A^i V)_{i=0,\cdots,2n/c}$ with $V = (v_1, \cdots, v_c)$, W two random matrices

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Parallelization over c machines :

$$1 \quad ({}^tWA^i v_1)_{i=0,\cdots,2n/c}$$

$$\cdots \qquad\qquad \cdots$$

$$c \quad ({}^tWA^i v_c)_{i=0,\cdots,2n/c}$$

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$\ldots \qquad \ldots$

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Complexity : $O(KN^2)$ but distributed over c machines.

3. Reconstruction of the minimal polynomial of $\boxed{A}$ thanks to Berlekamp-Massey algorithm.

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$$\cdots \qquad\qquad \cdots$$

$$c \quad ({}^t WA^i v_c)_{i=0,\cdots,2n/c}$$

   Complexity : $O(KN^2)$ but distributed over c machines.

3. Reconstruction of coeffs. a_{ij} s.t. $\sum_{j=1}^{c} \sum_{i=0}^{n/c} a_{ij} A^i v_j = 0$ thanks to Thomé algorithm. Complexity : $\tilde{O}(c^2 N)$
4. Computation of the solution.

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$c \quad ({}^tWA^i v_c)_{i=0,\cdots,2n/c}$

   Complexity : $\mathrm{O}(KN^2)$ but distributed over c machines.

3. Reconstruction of coeffs. a_{ij} s.t. $\sum_{j=1}^{c}\sum_{i=0}^{n/c} a_{ij}A^i v_j = 0$ thanks to Thomé algorithm. Complexity : $\tilde{O}(c^2N)$
4. Computation of the solution. Complexity : $O(KN^2)$ distributed.

From Wiedemann to Block Widemann

1. Preconditioning step: Transformation of $\boxed{S}$ into a square matrix $\boxed{A}$. The problem becomes:

$$A.x = 0 \quad \text{ou} \quad A.x = y'.$$

2. Computation of a matrix sequence: $({}^tW\ A^iV)_{i=0,\cdots,2n/c}$ with $V = (v_1, \cdots, v_c)$, W two random matrices
Parallelization over c machines :

$$1 \quad ({}^tWA^iv_1)_{i=0,\cdots,2n/c}$$

$$\cdots \qquad \cdots$$

$$c \quad ({}^tWA^iv_c)_{i=0,\cdots,2n/c}$$

Complexity : $O(KN^2)$ but distributed over c machines.

3. Reconstruction of coeffs. a_{ij} s.t. $\sum_{j=1}^{c}\sum_{i=0}^{n/c} a_{ij}A^iv_j = 0$ thanks to Thomé algorithm. Complexity : $\tilde{O}(c^2N)$
4. Computation of the solution. Complexity : $O(KN^2)$ distributed.

Final asymptotic complexity: $O(KN^2) + \tilde{O}(c^2N)$

Dlog-NFS raises a question of identity...

... what if the matrix is not *truly* sparse?

Matrices in NFS

Computing Dlog with NFS leads to consider matrices of the form:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 5 & 3 \\ 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & 6 & 4 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 5 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 6 \\ 0 & 3 & 0 & 1 & 0 & 0 & 2 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 4 & 2 \\ 0 & 2 & 0 & 3 & 0 & 0 & 2 & 0 & 5 & 1 \end{pmatrix}$$

Is it sparse? Is it dense?

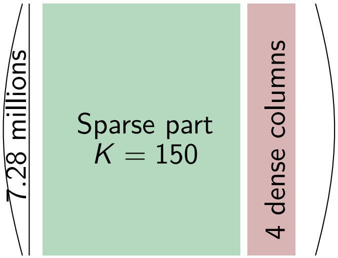
$K = 5$
$N = 11$

- If we apply a classical algo., we don't take advantage of zero coeffs.
- If we apply Block-Wiedemann, we don't take advantage of the particular distribution of non zero coeffs.

- Number fields complicate the linear algebra step: need to take into account the contribution of units in these number fields.
- $\Rightarrow$ Schirokauer maps.
- 1 unit $=$ +1 Schirokauer map $=$ +1 dense column

Example

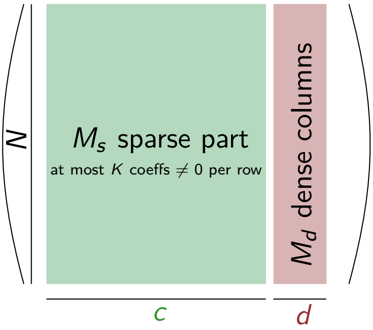
- Latest record on a prime field $\mathbb{F}_p$, ($p \approx 180$ digits)
- June 2014 by Bouvier, Gaudry, Imbert, Jeljeli, Thomé.



Nearly sparse linear algebra

Definition

M is (d-)nearly sparse if it is of the form:



Problem

Solve: $M \cdot x = 0$ *or* $M \cdot x = y$

where M is a nearly sparse matrix with coeff. in a ring $\mathbb{K}$.

Nearly sparse linear algebra

Remark

- There is no restriction on the nbr of dense columns.

Nearly sparse linear algebra

Remark

- There is no restriction on the nbr of dense columns.
- Being able to recover a non trivial elt of the kernel of a nearly sparse matrix suffices!

Let's assume we want to solve $M \cdot x = y$ with M a d-nearly sparse matrix.

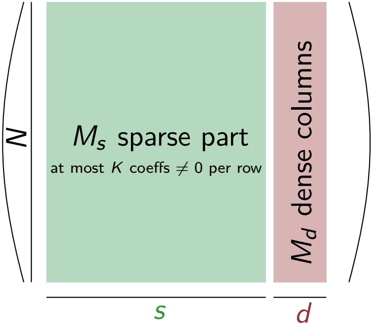
Then $\begin{pmatrix} M \end{pmatrix} \cdot \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} y \end{pmatrix} \Leftrightarrow \begin{pmatrix} M & y \end{pmatrix} \cdot \begin{pmatrix} x \\ -1 \end{pmatrix} = 0.$

Since $\begin{pmatrix} M & y \end{pmatrix}$ is $d+1$-nearly sparse, it's ok.

Nearly sparse linear algebra

Definition

M is (d-)nearly sparse if it is of the form:



Problem

Solve:

$$M \cdot x = 0$$

where M is a nearly sparse matrix with coeff. in a ring $\mathbb{K}$.

A dedicated algorithm

Since M is (also) a sparse matrix of parameters $K+d$, N, we may apply Block-Wiedemann!
Asymptotic complexity:

$$O((K+d)N^2) + \tilde{O}(c^2 N)$$

A dedicated algorithm

Since M is (also) a sparse matrix of parameters $K+d$, N, we may apply Block-Wiedemann!
Asymptotic complexity:

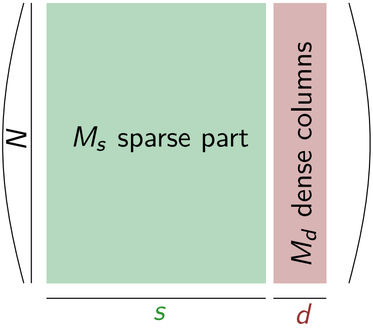
$$O((K+d)N^2)+\tilde{O}(c^2N)$$

Main result

We propose to design an algorithm with asymptotic complexity:

$$O(KN^2)+\tilde{O}(\max(c^2,d^2)N)$$

Key ideas



1. Apply Block-Wiedemann on the sparse part only.
2. Make the d dense columns contribute in the initial block V, *i.e.* set each dense col. $=$ one initial vector of the matrix sequences to construct.

Nearly sparse linear algebra algorithm

1. Preconditioning step on the RIGHT of the matrix M :

$$\begin{pmatrix} M_s & M_d \end{pmatrix} \times \begin{array}{c|cc} & s+d & d \\ \hline s & R & (0) \\ d & (0) & I_d \end{array} = \begin{pmatrix} A & M_d \end{pmatrix}$$

Why ?

- Powers of A are well defined.
- Multiplying $M_s \; R = A$ by a vector is quick enough.
- If R surj. : $(A|M_d).x = 0 \Rightarrow M.x = 0$

Try to solve $(A|M_d).x = 0$.

Nearly sparse linear algebra algorithm

For the sake of simplicity: # machines = # dense col.

2. Computation of a matrix sequence: $({}^tWA^iV)_{i=0,\cdots,2N}$
   with $V = (v_1, \cdots, v_d)$, W two rand. matrices.
   Parallelization over c machines :

1 $({}^tWA^iv_1)_{i=0,\cdots,2N/d}$

$\cdots \qquad \cdots$

d $({}^tWA^iv_d)_{i=0,\cdots,2N/d}$

.

Nearly sparse linear algebra algorithm

For the sake of simplicity: # machines = # dense col.

2. Computation of a matrix sequence: $({}^tWA^iV)_{i=0,\cdots,2N}$ with $V = (d_1, \cdots, d_d)$, W one rand. matrix and $d_1, \cdots, d_d$ the d dense col.
   Parallelization over d machines :

$1 \quad ({}^tWA^id_1)_{i=0,\cdots,2N/d}$

$\cdots \qquad \cdots$

$d \quad ({}^tWA^id_d)_{i=0,\cdots,2N/d}$

.

Nearly sparse linear algebra algorithm

For the sake of simplicity: # machines = # dense col.

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$\cdots \qquad \cdots$

$d \quad ({}^t W A^i d_d)_{i=0,\cdots,2N/d}$

3. Reconstruction of coeffs. a_{ij} s.t. $\sum_{j=1}^{d} \sum_{i=0}^{N/d} a_{ij} A^i d_j = 0$
   thanks to Thomé.

Nearly sparse linear algebra algorithm

4. Computation of an elt of the kernel of $\begin{pmatrix} A & M_d \end{pmatrix}$

$$\sum_{j=1}^{d}\sum_{i=0}^{N/d} a_{ij}A^i d_j = 0 \Leftrightarrow \sum_{j=1}^{d}\sum_{i=1}^{N/d} a_{ij}A^i d_j + \sum_{j=1}^{d} a_{0j} d_j = 0$$

$$\Leftrightarrow A\cdot \underbrace{\sum_{j=1}^{d}\sum_{i=1}^{N/d} a_{ij}A^{i-1} d_j}_{\text{let us say } x'} + \sum_{j=1}^{d} a_{0j} d_j = 0$$

$$\Leftrightarrow A\cdot x' + a_{01} d_1 + a_{02} d_2 + \cdots + a_{0d} d_d = 0$$

So ${}^t(x'|a_{01}|a_{02}|\cdots|a_{0d}) \in \ker \begin{pmatrix} A & M_d \end{pmatrix}$.

Asymptotic complexity

Main result

We obtain an asymptotic complexity of:

$$O(KN^2) + \tilde{O}(\max(c^2, d^2)N) \textit{ operations,}$$

to be compared with previous $O((K + d)N^2) + \tilde{O}(c^2N)$ *complexity.*

When $d \leq c$, it becomes:

$$O(KN^2) + \tilde{O}(c^2N) \text{ operations.}$$

Remark

When we have more machines than dense columns, these columns cost NOTHING with our algorithm!

Asymptotic Complexity

And if $c < d$, how many dense col. can we still have?

- As soon as $d < N^{1-\epsilon}$ $(\epsilon > 0)$, our algorithm is better than Block-Wiedemann.
- As soon as $d < N^{\omega-2-\epsilon}$ $(\epsilon > 0)$, it is better than classical (dense) linear algebra algorithms of complexity $O(N^{\omega})$.

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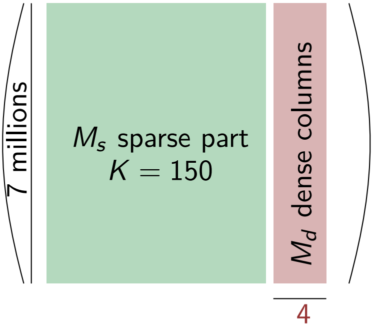
Example

Recalling that $\omega \approx 2.37$, with $N^{1/3}$ dense columns for instance, our algorithm is still faster than any others.

Nearly Sparse Linear Algebra applied to Dlog

- Latest record on a prime field $\mathbb{F}_p$, ($p \approx 180$ digits)
- June 2014 by Bouvier, Gaudry, Imbert, Jeljeli, Thomé.
- Parameters of the matrix: $N \approx 7, 28$ millions of rows, $K = 150$ non zero coeff. per row, 4 dense columns.
- Parallelized over 16 machines.



To conclude with medium characteristic

- If your are a cryptographer:
  increase your finite fields cardinality by 6.4%
- If you are a cryptanalyst:
  do not worry about dense columns.

Merci de votre attention !