ECC Summer School, Bordeaux, France — September 23–25, 2015

Software and Hardware Implementation of Elliptic Curve Cryptography

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Context: Elliptic curves

- Let us consider a finite field $\mathbb{F}_q$ and an elliptic curve $E/\mathbb{F}_q$

  e.g., $E : y^2 = x^3 + Ax + B$, with parameters $A, B \in \mathbb{F}_q$ and $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$

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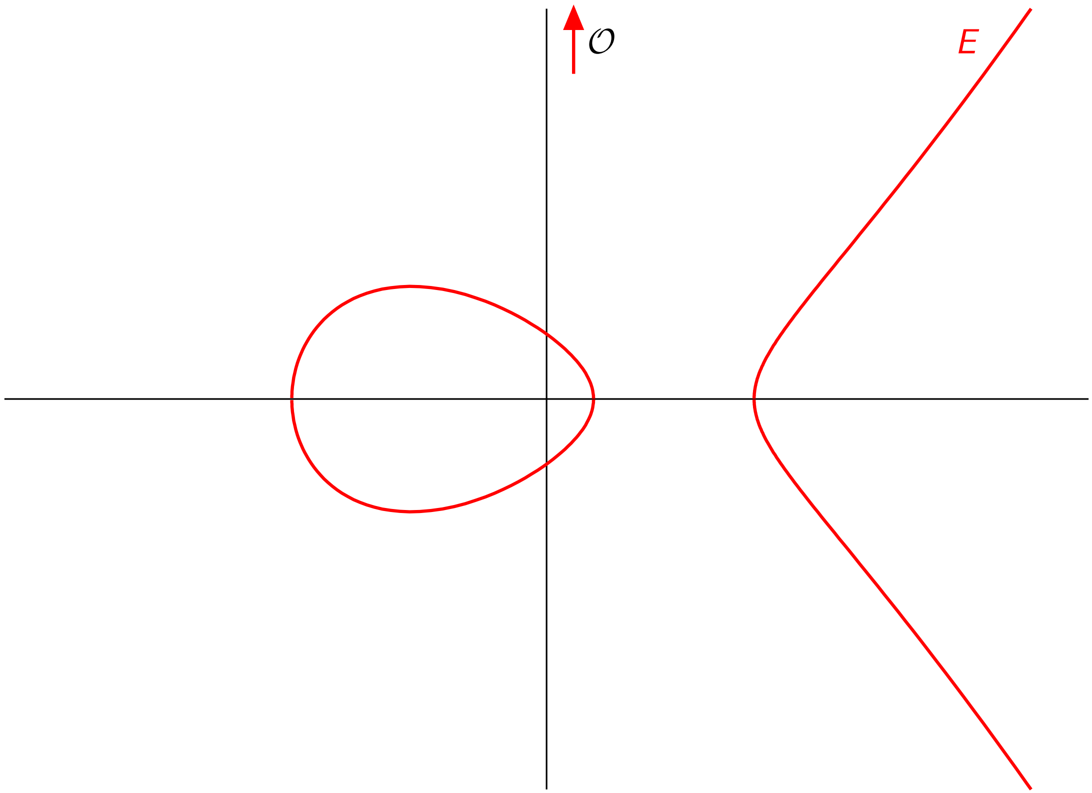
- Additive group law: $E(\mathbb{F}_q)$ is an abelian group
  - addition via the "chord and tangent" method
  - $\mathcal{O}$ is the neutral element

  [See D. Robert's lectures]

The group law



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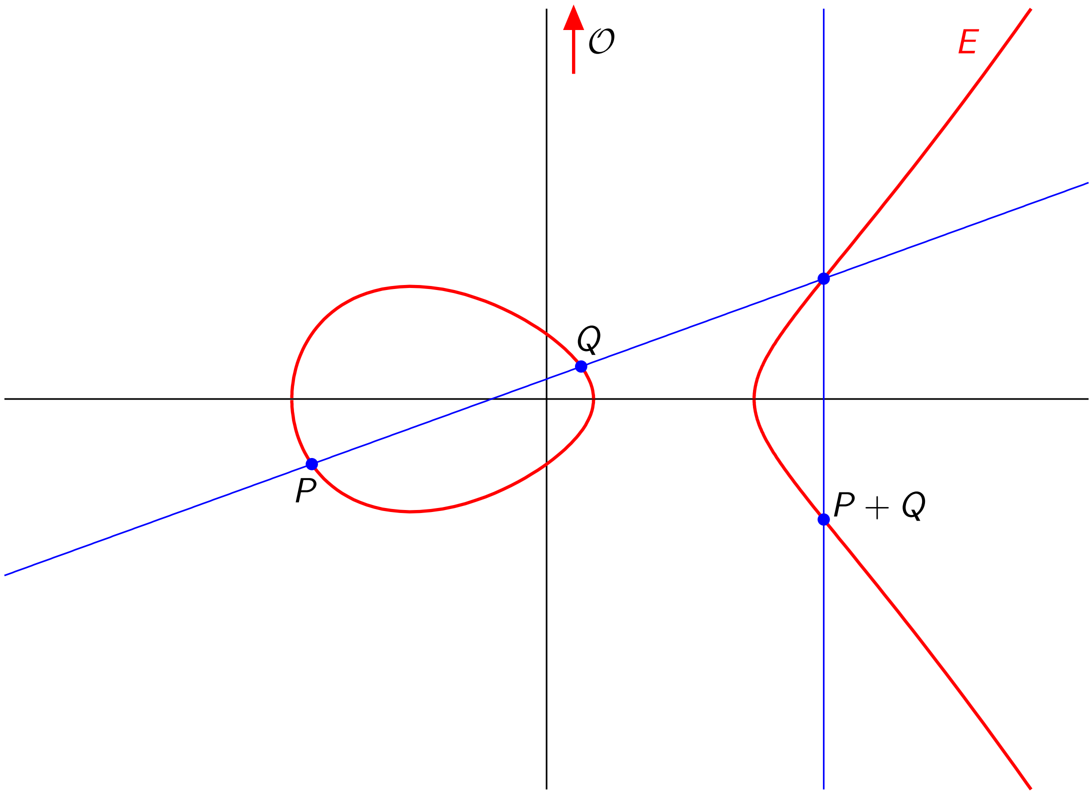

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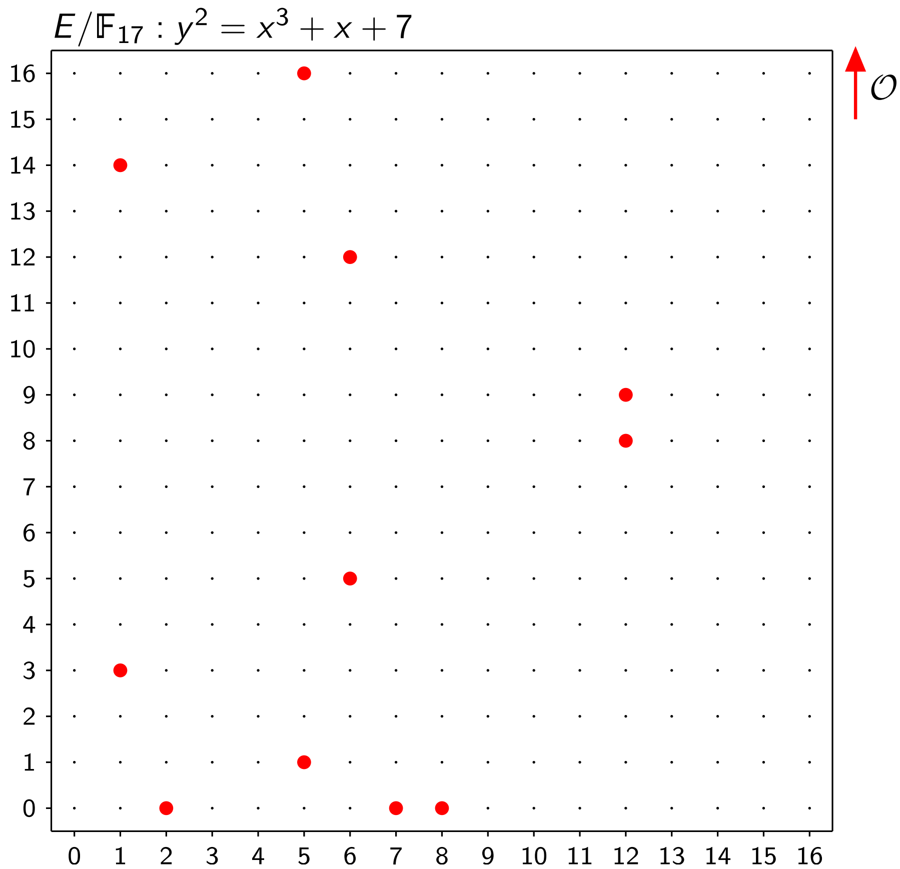

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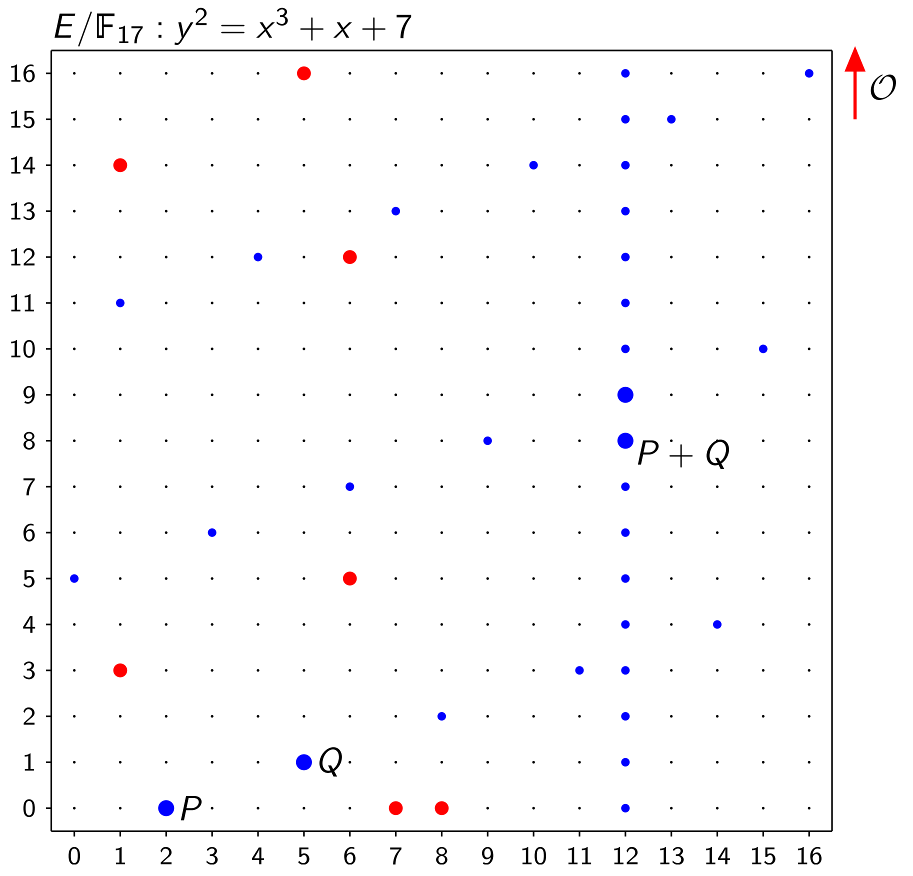

Scalar multiplication and discrete logarithm

- $E(\mathbb{F}_q)$ is a finite abelian group:
  - let $\mathbb{G}$ be a cyclic subgroup of $E(\mathbb{F}_q)$
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- The inverse map is the so-called discrete logarithm (in base P):

$$\begin{array}{rccl} \mathrm{dlog}_P = \exp_P^{-1} : & \mathbb{G} & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} \\ & Q & \longmapsto & k \qquad \text{such that } Q = kP \end{array}$$

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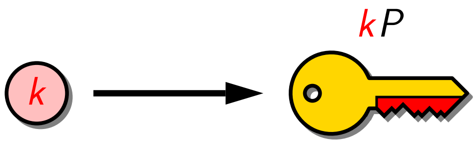

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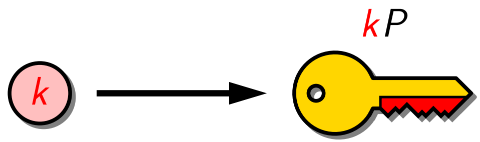

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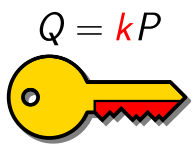

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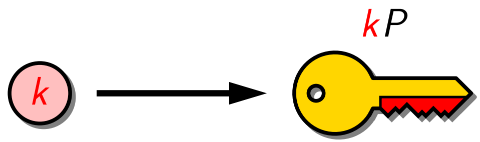

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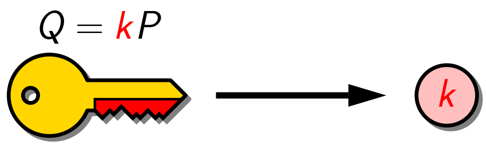

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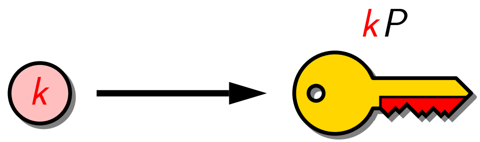

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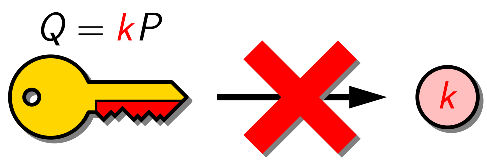

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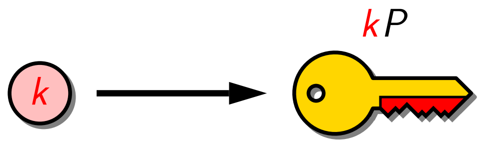

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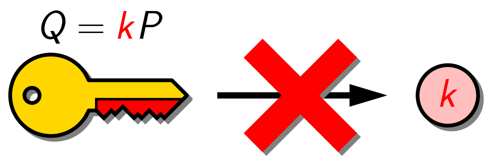

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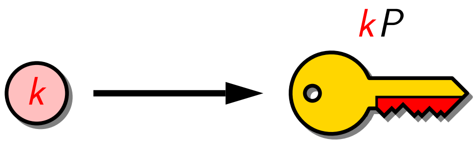

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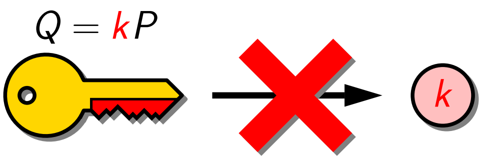

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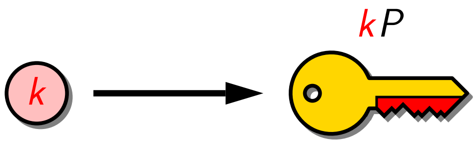

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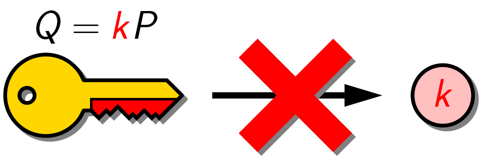

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- That's a one-way function $\Rightarrow$ Public-key cryptography!
  - private key: an integer k in $\mathbb{Z}/\ell\mathbb{Z}$
  - public key: the point kP in $\mathbb{G} \subseteq E(\mathbb{F}_q)$

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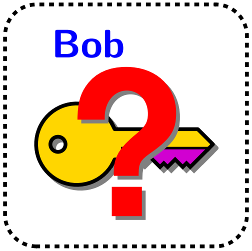

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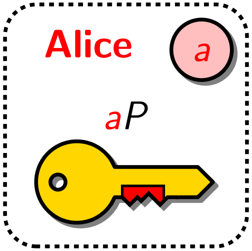

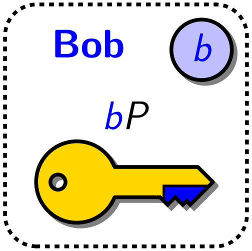

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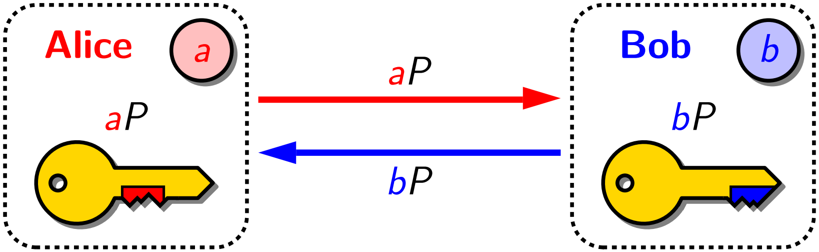

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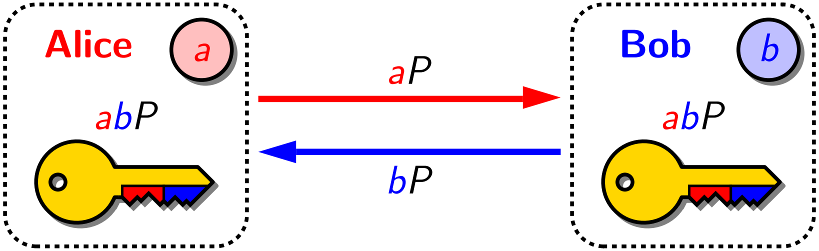

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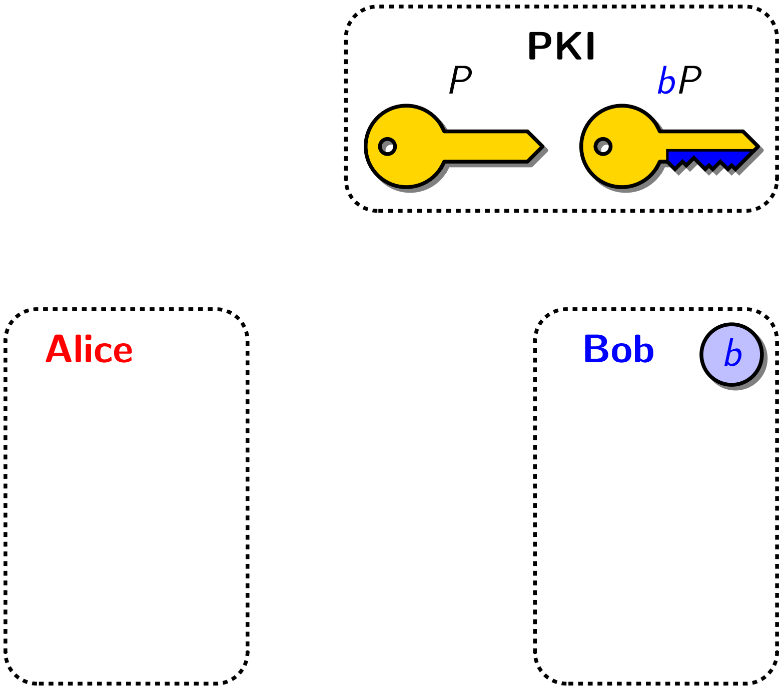

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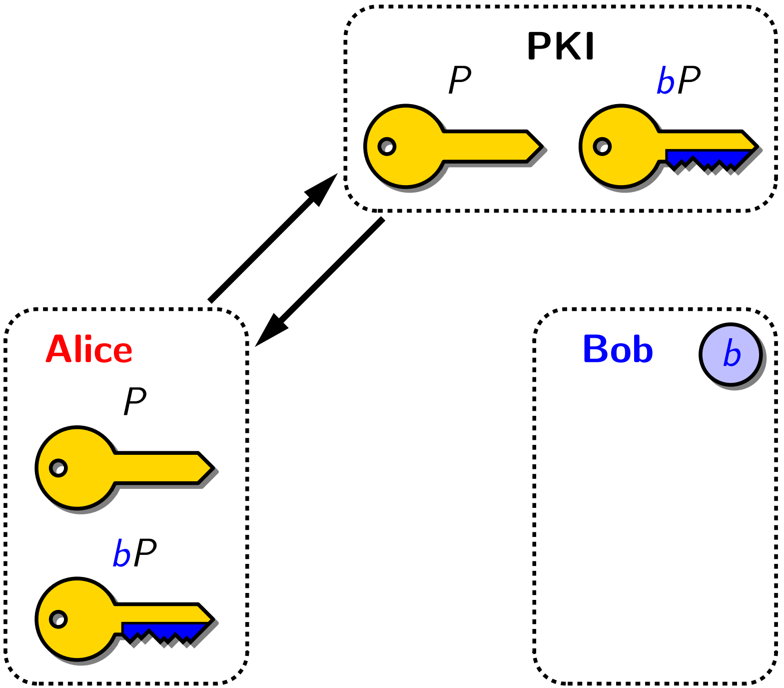

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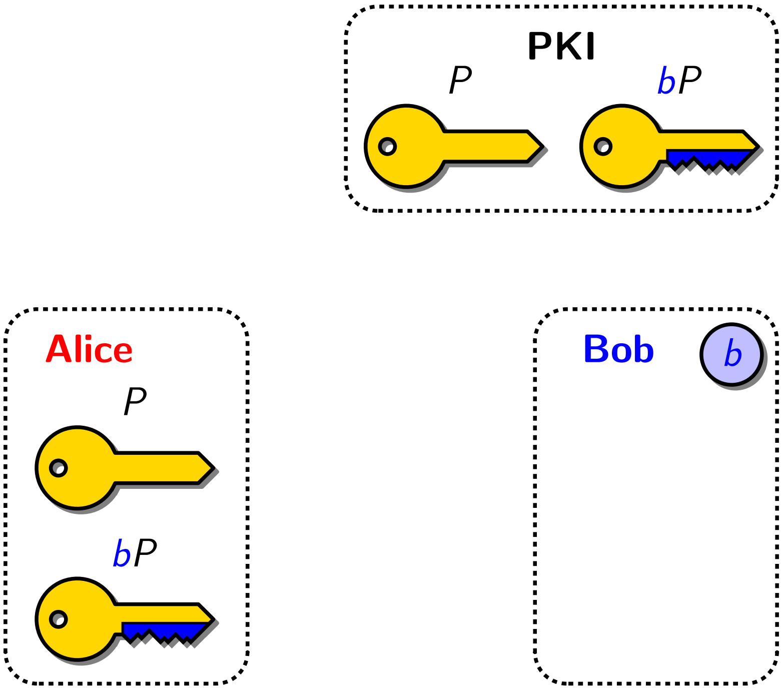

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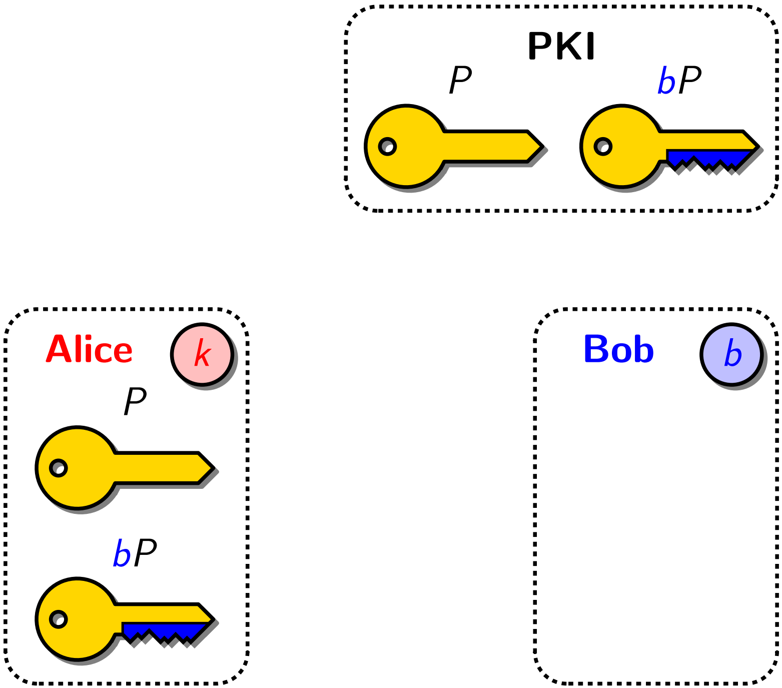

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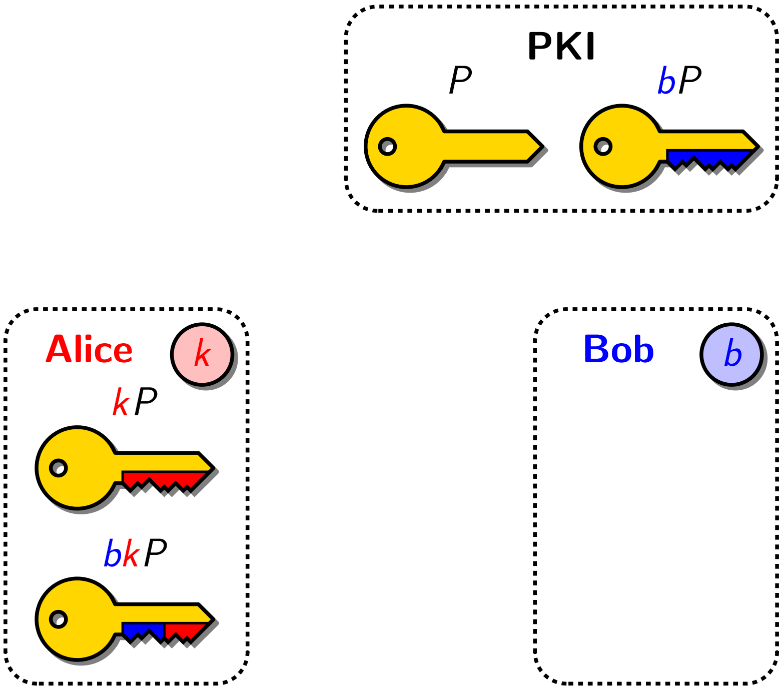

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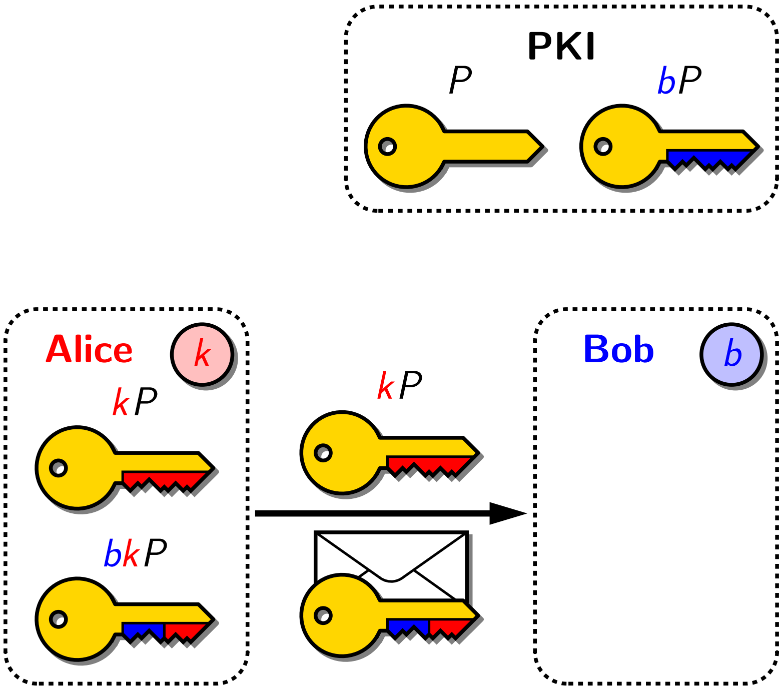

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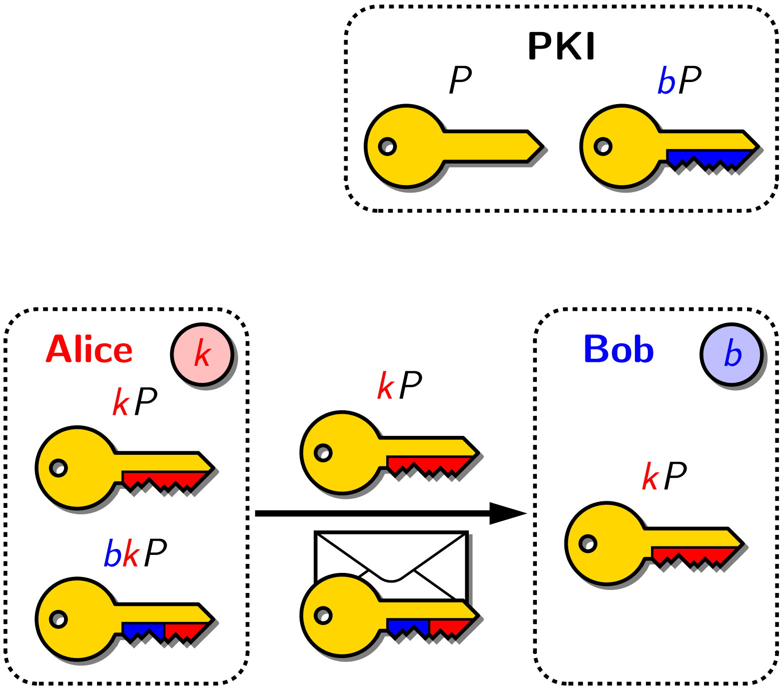

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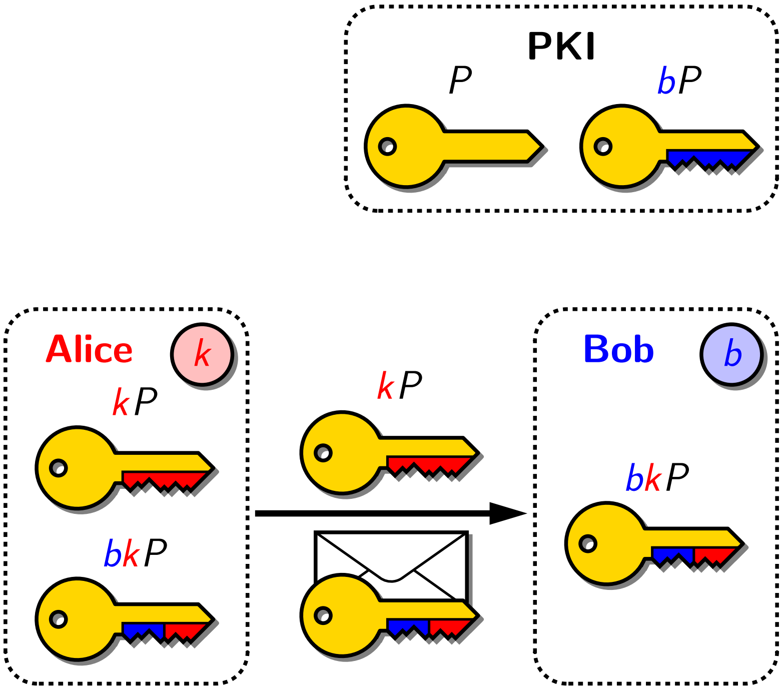

Central operation: the scalar multiplication

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- Other important operations might be required, such as pairings [See J. Krämer's talk]

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- Other possible target platforms, mostly for cryptanalytic computations:
  - clusters of CPUs
  - GPUs (graphics processors)
  - FPGAs (reconfigurable circuits)

Which target platforms?

- Cryptography should be available everywhere:
  - on desktop PCs and laptops
    → 64-bit Intel or AMD CPUs with SIMD instructions (SSE / AVX)
  - on smartphones
    → low-power 32- or 64-bit ARM CPUs, maybe with SIMD (NEON)
  - on wireless sensors
    → tiny 8-bit microcontroller (such as Atmel AVRs)
  - on smart cards and RFID chips
    → custom cryptoprocessor (ASIC or ASIP) with dedicated hardware for cryptographic operations
- Other possible target platforms, mostly for cryptanalytic computations:
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  - GPUs (graphics processors)
  - FPGAs (reconfigurable circuits)

  ⇒ In such cases, implementation security is usually less critical

Implementation layers

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  $\Rightarrow$ ECC is no more secure than its weakest link

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- In these lectures, we will mostly focus on the **green layers**

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- There already exist several free-software, open-source implementations of ECC (or of useful layers thereof):

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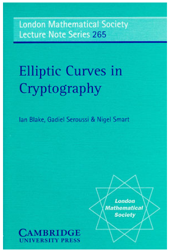
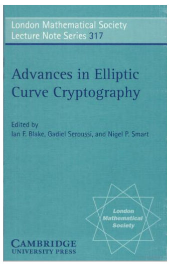
  - PAVOIS project (announced) [See A. Tisserand's talk]

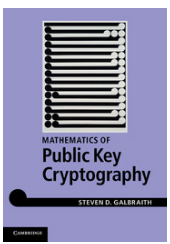
Some references



Elliptic Curves in Cryptography,
Ian F. Blake, Gadiel Seroussi, and Nigel P. Smart.
London Mathematical Society 265,
Cambridge University Press, 1999.

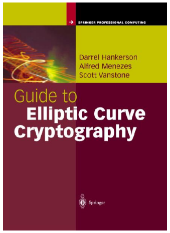


Advances in Elliptic Curves Cryptography,
Ian F. Blake, Gadiel Seroussi, and Nigel P. Smart (editors).
London Mathematical Society 317,
Cambridge University Press, 2005.

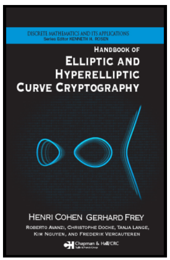


Mathematics of Public-Key Cryptography,
Steven D. Galbraith.
Cambridge University Press, 2012.

Some references



Guide to Elliptic Curve Cryptography,
Darrel Hankerson, Alfred Menezes, and Scott Vanstone.
Springer, 2004.



Handbook of Elliptic and Hyperelliptic Curve Cryptography,
Henri Cohen and Gerhard Frey (editors).
Chapman & Hall / CRC, 2005.

Proceedings of the CHES workshop and of other crypto conferences.

Outline

I. Scalar multiplication

II. Elliptic curve arithmetic

III. Finite field arithmetic

IV. Software considerations

V. Notions of hardware design

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Scalar multiplication

- Given k in $\mathbb{Z}/\ell\mathbb{Z}$ and P in $\mathbb{G} \subseteq E(\mathbb{F}_q)$, we want to compute

$$kP = \underbrace{P + P + \ldots + P}_{k \text{ times}}$$

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- Size of ℓ (and k) for crypto applications: between 250 and 500 bits
- Repeated addition, in $O(k)$ complexity, is out of the question!

Double-and-add algorithm

- Available operations on $E(\mathbb{F}_q)$:
  - point addition: $(Q, R) \mapsto Q + R$
  - point doubling: $Q \mapsto 2Q = Q + Q$

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  - start from the most significant bit of k
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- Idea: iterative algorithm based on the binary expansion of k
  - start from the most significant bit of k
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  - add P if the corresponding bit of k is 1
  - same principle as binary exponentiation

Double-and-add algorithm

- Denoting by $(k_{n-1} \ldots k_1 k_0)_2$, with $n = \lceil \log_2 \ell \rceil$, the binary expansion of k:

```
function scalar-mult(k, P):
    T ← O
    for i ← n − 1 downto 0:
        T ← 2T
        if k_i = 1:
            T ← T + P
    return T
```

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- Example: $k = 431$

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$$T = \qquad = \mathcal{O}$$

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    return T

- Example: $k = 431 = (\underline{1}10101111)_2$

$T = \quad P \quad\quad = \quad P$

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$$
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\quad T \leftarrow \mathcal{O} \\
\quad \textbf{for } i \leftarrow n-1 \textbf{ downto } 0\textbf{:} \\
\quad\quad T \leftarrow 2T \\
\quad\quad \textbf{if } k_i = 1\textbf{:} \\
\quad\quad\quad T \leftarrow T + P \\
\quad \textbf{return } T
\end{array}
$$

- Example: $k = 431 = (1\underline{1}0101111)_2$

$$T = P \cdot 2 = 2P$$

Double-and-add algorithm

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  return T

- Example: $k = 431 = (1\underline{1}0101111)_2$

$$T = \quad P \cdot 2 + P \quad = \quad 3P$$

Double-and-add algorithm

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        if $k_i = 1$**:**
            $T \leftarrow T + P$
    return T

- Example: $k = 431 = (11\underline{0}101111)_2$

$$T = (P \cdot 2 + P) \cdot 2 = 6P$$

Double-and-add algorithm

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\quad\quad\quad T \leftarrow T + P \\
\quad \textbf{return } T
\end{array}
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- Example: $k = 431 = (110\underline{1}01111)_2$

$$T = (P \cdot 2 + P) \cdot 2^2 = 12P$$

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- Example: $k = 431 = (1101\underline{0}1111)_2$

$$T = ((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2 = 26P$$

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- Example: $k = 431 = (110101\underline{1}11)_2$

$$T = \quad ((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 \quad = \quad 52P$$

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- Example: $k = 431 = (110101\underline{1}11)_2$

$$T = (((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P) \cdot 2 \qquad = 106P$$

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- Example: $k = 431 = (110101111)_2$

$$T = ((((P \cdot 2 + P) \cdot 2^2 + P) \cdot 2^2 + P) \cdot 2 + P) \cdot 2 \qquad = 214P$$

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- Example: $k = 431 = (11010111\underline{1})_2$

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- Complexity in $O(n) = O(\log_2 \ell)$ operations over $E(\mathbb{F}_q)$:
  - n doublings, and
  - $n/2$ additions on average

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks

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- Precompute $2P$, $3P$, $\ldots$, $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, ..., $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
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- Example with $w = 3$: $k = 431$

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2$

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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- Precompute $2P$, $3P$, ..., $(2^w - 1)P$:
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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = \qquad\qquad\qquad = \quad \mathcal{O}$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
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  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (\underline{110}\,101\,111)_2 = (\underline{6}57)_{2^3}$

$$T = 6P \qquad = \quad 6P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, ..., $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,\underline{101}\,111)_2 = (6\underline{5}7)_{2^3}$

$$T = 6P \cdot 2^3 \qquad = 48P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, $\ldots$, $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,\underline{101}\,111)_2 = (6\underline{5}7)_{2^3}$

$$T = 6P \cdot 2^3 + 5P \qquad = 53P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, ..., $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,101\,\underline{111})_2 = (65\underline{7})_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 \qquad = 424P$$

Windowed method

- Consider 2^w-ary expansion of k: i.e., split k into w-bit chunks
- Precompute $2P$, $3P$, $\ldots$, $(2^w - 1)P$:
  - $2^{w-1} - 1$ doublings, and
  - $2^{w-1} - 1$ additions
- Example with $w = 3$: $k = 431 = (110\,101\,\underline{111})_2 = (65\underline{7})_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

Windowed method

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- Precompute $2P$, $3P$, ..., $(2^w - 1)P$:
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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

- Complexity:
  - n doublings, and
  - $(1 - 2^{-w})n/w$ additions on average

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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

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- Select w carefully so that precomputation cost does not become predominant

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  - $2^{w-1} - 1$ doublings, and
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- Example with $w = 3$: $k = 431 = (110\,101\,111)_2 = (657)_{2^3}$

$$T = (6P \cdot 2^3 + 5P) \cdot 2^3 + 7P = 431P$$

- Complexity:
  - n doublings, and
  - $(1 - 2^{-w})n/w$ additions on average
- Select w carefully so that precomputation cost does not become predominant
- Sliding window variant: half as many precomputations

Non-adjacent form

- Fact: computing the opposite of a point on $E(\mathbb{F}_q)$ has a negligible cost

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- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431$

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  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w=3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

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  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

$$T = \qquad = \quad \mathcal{O}$$

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- Precompute $3P$, $5P$, $\ldots$, $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (\underline{3}003000\bar{1})_2$

$$T = 3P \qquad = \quad 3P$$

Non-adjacent form

- Fact: computing the opposite of a point on $E(\mathbb{F}_q)$ has a negligible cost
- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (300300 0\bar{1})_2$

$$T = 3P \cdot 2 \quad = \quad 6P$$

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- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (300\underline{0}3000\bar{1})_2$

$$T = 3P \cdot 2^2 = 12P$$

Non-adjacent form

- Fact: computing the opposite of a point on $E(\mathbb{F}_q)$ has a negligible cost
- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

$$T = 3P \cdot 2^3 \qquad = 24P$$

Non-adjacent form

- Fact: computing the opposite of a point on $E(\mathbb{F}_q)$ has a negligible cost
- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (300\underline{3}000\bar{1})_2$

$$T = 3P \cdot 2^3 + 3P \qquad = 27P$$

Non-adjacent form

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- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, $\ldots$, $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003\underline{0}00\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2 \qquad = \quad 54P$$

Non-adjacent form

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- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, …, $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w=3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^2 \qquad = 108P$$

Non-adjacent form

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- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \dots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\underline{0}\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^3 \qquad = 216P$$

Non-adjacent form

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- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, $\ldots$, $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (300300\underline{0}\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^4 \qquad = 432P$$

Non-adjacent form

- Fact: computing the opposite of a point on $E(\mathbb{F}_q)$ has a negligible cost
- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
- Precompute $3P$, $5P$, ..., $(2^{w-1}-1)P$:
  - 1 doubling, and
  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\underline{\bar{1}})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^4 - P = 431P$$

Non-adjacent form

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- Idea: use signed digits to represent scalar k with minimal Hamming weight
- 2^w-ary non-adjacent form (w-NAF): use odd digits $\{-2^{w-1}+1, \ldots, 2^{w-1}-1\}$ and 0 to represent k so that at most every w-th digit is non-zero
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  - $2^{w-2}-1$ additions
- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^4 - P = 431P$$

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- Example with $w = 3$ (digits in $\{\bar{3}, \bar{1}, 0, 1, 3\}$): $k = 431 = (3003000\bar{1})_2$

$$T = (3P \cdot 2^3 + 3P) \cdot 2^4 - P = 431P$$

- Complexity:
  - n doublings, and
  - $n/(w+1)$ additions on average

Multi-exponentiation technique

- To compute the sum of several scalar multiplications

$$\text{e.g., } aP + bQ, \text{ where } a, b \in \mathbb{Z}/\ell\mathbb{Z} \text{ and } P, Q \in E(\mathbb{F}_q)$$

Multi-exponentiation technique

- To compute the sum of several scalar multiplications

$$\text{e.g., } aP + bQ, \text{ where } a, b \in \mathbb{Z}/\ell\mathbb{Z} \text{ and } P, Q \in E(\mathbb{F}_q)$$

- Idea:
  - compute and accumulate all scalar multiplications simultaneously
  - share doubling steps between multiplications

function double-scalar-mult(a, P, b, Q)**:**
- $S \leftarrow P + Q$
- $T \leftarrow \mathcal{O}$
- **for** $i \leftarrow n - 1$ **downto** 0**:**
  - $T \leftarrow 2T$
  - **if** $a_i = 1$ **and** $b_i = 1$**:**
    - $T \leftarrow T + S$
  - **else if** $a_i = 1$**:**
    - $T \leftarrow T + P$
  - **else if** $b_i = 1$**:**
    - $T \leftarrow T + Q$
- **return** T

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**

```
S ← P + Q
T ← O
for i ← n − 1 downto 0:
    T ← 2T
    if a_i = 1 and b_i = 1:
        T ← T + S
    else if a_i = 1:
        T ← T + P
    else if b_i = 1:
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return T
```

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        T ← 2T
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            T ← T + S
        else if a_i = 1:
            T ← T + P
        else if b_i = 1:
            T ← T + Q
    return T
```

- Example: $a = 21$
  and $b = 30$

Multi-exponentiation technique

```
function double-scalar-mult(a, P, b, Q):
    S ← P + Q
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            T ← T + S
        else if a_i = 1:
            T ← T + P
        else if b_i = 1:
            T ← T + Q
    return T
```

- Example: $a = 21 = (10101)_2$
  and $b = 30 = (11110)_2$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**
$S \leftarrow P + Q$
$T \leftarrow \mathcal{O}$
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else if $b_i = 1$**:**
$T \leftarrow T + Q$
return T

- Example: $a = 21 = (10101)_2$
  and $b = 30 = (11110)_2$

$T = \qquad = \qquad \mathcal{O}$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**
    $S \leftarrow P + Q$
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        else if $a_i = 1$**:**
            $T \leftarrow T + P$
        else if $b_i = 1$**:**
            $T \leftarrow T + Q$
    return T

- Example: $a = 21 = (\underline{1}0101)_2$
  and $b = 30 = (\underline{1}1110)_2$

$T = \quad P + Q \quad\quad = \quad P + \quad Q$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**

$S \leftarrow P + Q$
$T \leftarrow \mathcal{O}$
for $i \leftarrow n-1$ **downto** 0**:**
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  if $a_i = 1$ **and** $b_i = 1$**:**
    $T \leftarrow T + S$
  else if $a_i = 1$**:**
    $T \leftarrow T + P$
  else if $b_i = 1$**:**
    $T \leftarrow T + Q$
return T

- Example: $a = 21 = (1\underline{0}101)_2$
  and $b = 30 = (1\underline{1}110)_2$

$$T = \quad (P + Q) \cdot 2 \qquad = \; 2P + \; 2Q$$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**

```
S ← P + Q
T ← O
for i ← n − 1 downto 0:
    T ← 2T
    if a_i = 1 and b_i = 1:
        T ← T + S
    else if a_i = 1:
        T ← T + P
    else if b_i = 1:
        T ← T + Q
return T
```

- Example: $a = 21 = (1\underline{0}101)_2$
  and $b = 30 = (1\underline{1}110)_2$

$$T = (P + Q) \cdot 2 + Q \qquad = 2P + 3Q$$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**

$S \leftarrow P + Q$
$T \leftarrow \mathcal{O}$
for $i \leftarrow n - 1$ **downto** 0**:**
  $T \leftarrow 2T$
  if $a_i = 1$ **and** $b_i = 1$**:**
    $T \leftarrow T + S$
  else if $a_i = 1$**:**
    $T \leftarrow T + P$
  else if $b_i = 1$**:**
    $T \leftarrow T + Q$
return T

- Example: $a = 21 = (10\underline{1}01)_2$
  and $b = 30 = (11\underline{1}10)_2$

$$T = ((P + Q) \cdot 2 + Q) \cdot 2 \qquad = 4P + 6Q$$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q):
- $S \leftarrow P + Q$
- $T \leftarrow \mathcal{O}$
- **for** $i \leftarrow n-1$ **downto** 0:
  - $T \leftarrow 2T$
  - **if** $a_i = 1$ **and** $b_i = 1$:
    - $T \leftarrow T + S$
  - **else if** $a_i = 1$:
    - $T \leftarrow T + P$
  - **else if** $b_i = 1$:
    - $T \leftarrow T + Q$
- **return** T

- Example: $a = 21 = (10\underline{1}01)_2$ and $b = 30 = (11\underline{1}10)_2$

$$T = ((P+Q)\cdot 2 + Q)\cdot 2 + P + Q \qquad = 5P + 7Q$$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**

$S \leftarrow P + Q$

$T \leftarrow \mathcal{O}$

for $i \leftarrow n-1$ **downto** 0**:**

$T \leftarrow 2T$

if $a_i = 1$ **and** $b_i = 1$**:**

$T \leftarrow T + S$

else if $a_i = 1$**:**

$T \leftarrow T + P$

else if $b_i = 1$**:**

$T \leftarrow T + Q$

return T

- Example: $a = 21 = (10101)_2$
  and $b = 30 = (11110)_2$

$$T = (((P + Q) \cdot 2 + Q) \cdot 2 + P + Q) \cdot 2 \qquad = 10P + 14Q$$

Multi-exponentiation technique

function double-scalar-mult(a, P, b, Q)**:**
  $S \leftarrow P + Q$
  $T \leftarrow \mathcal{O}$
  for $i \leftarrow n - 1$ **downto** 0**:**
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- Example: $a = 21 = (1010\underline{1}1)_2$ and $b = 30 = (111\underline{1}0)_2$

$$T = (((P + Q) \cdot 2 + Q) \cdot 2 + P + Q) \cdot 2 + Q \qquad = 10P + 15Q$$

Multi-exponentiation technique

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- With signed digits:
  - joint sparse form (JSF): $n/2$ additions
  - interleaved w-NAF: $2n/(w+1)$ additions

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  - then $\psi : (x, y) \mapsto (-x, \xi y)$ is an endomorphism of E and, since

$$\psi^2(x, y) = (x, -y) = -(x, y),$$

its characteristic polynomial is $\chi_\psi(T) = T^2 + 1$ and $\lambda = \pm\sqrt{-1} \bmod \ell$

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- Popular constructions exploiting endomorphism ring:
  - GLS curves (Galbraith, Lin, and Scott, 2008): large class of GLV-compatible curves
  - Koblitz curves: binary curves, with Frobenius map $\psi : (x, y) \mapsto (x^2, y^2)$

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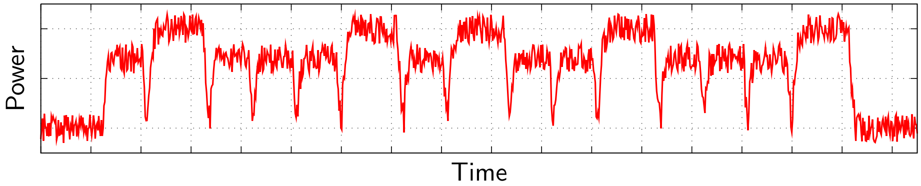

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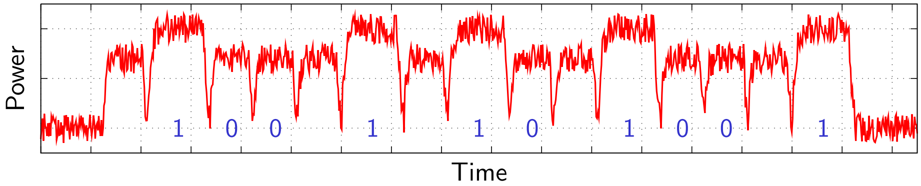

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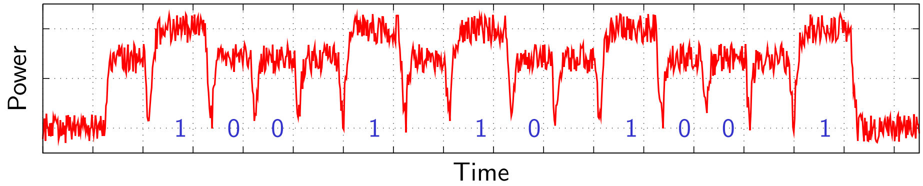

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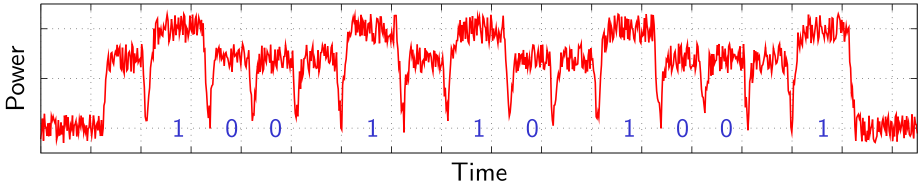

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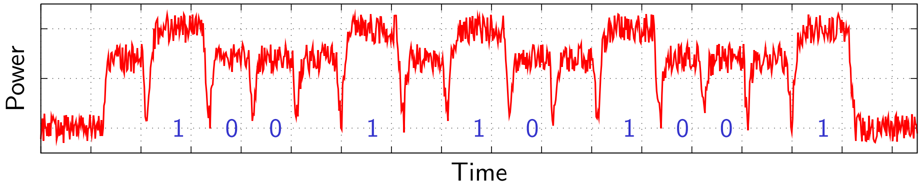
  - careful timing analysis will reveal Hamming weight of secret k
  - power analysis will leak bits of k



- Use double-and-add-always algorithm?
  - the result of the point addition is used if and only if $k_i = 1$

  $\Rightarrow$ vulnerable to fault attacks

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
$T_0 \leftarrow \mathcal{O}$
$T_1 \leftarrow P$
for $i \leftarrow n-1$ **downto** 0**:**
if $k_i = 1$**:**
$T_0 \leftarrow T_0 + T_1$
$T_1 \leftarrow 2T_1$
else:
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- Algorithm proposed by Montgomery in 1987:

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function scalar-mult(k, P):
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```

- Properties:

The Montgomery ladder

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```

- Properties:
  - perform one addition and one doubling at each step

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```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step

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```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

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return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
    $T_0 \leftarrow \mathcal{O}$
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            $T_0 \leftarrow 2T_0$
    return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10011)_2$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
            T_1 ← 2T_1
        else:
            T_1 ← T_0 + T_1
            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10011)_2$

$$T_0 = \qquad = \mathcal{O}$$
$$T_1 = P \qquad = P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
  $T_0 \leftarrow \mathcal{O}$
  $T_1 \leftarrow P$
  for $i \leftarrow n-1$ **downto** 0**:**
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    else:
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      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = \quad = \mathcal{O}$$
$$T_1 = P \quad = P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
$T_0 \leftarrow \mathcal{O}$
$T_1 \leftarrow P$
for $i \leftarrow n - 1$ **downto** 0**:**
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$T_0 \leftarrow T_0 + T_1$
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return T_0

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  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = P \quad = P$$
$$T_1 = P \quad = P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
    $T_0 \leftarrow \mathcal{O}$
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    for $i \leftarrow n - 1$ **downto** 0**:**
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    return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (\underline{1}0011)_2$

$$T_0 = P \qquad = P$$
$$T_1 = P \cdot 2 \qquad = 2P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
    T_1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T_0 ← T_0 + T_1
            T_1 ← 2T_1
        else:
            T_1 ← T_0 + T_1
            T_0 ← 2T_0
    return T_0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \quad = P$$
$$T_1 = P \cdot 2 \quad = 2P$$

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  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \quad = P$$
$$T_1 = P \cdot 2 + P \quad = 3P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
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      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$

- Example: $k = 19 = (1\underline{0}011)_2$

$$T_0 = P \cdot 2 = 2P$$
$$T_1 = P \cdot 2 + P = 3P$$

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      $T_1 \leftarrow T_0 + T_1$
      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2 = 2P$$
$$T_1 = P \cdot 2 + P = 3P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T0 ← O
    T1 ← P
    for i ← n − 1 downto 0:
        if k_i = 1:
            T0 ← T0 + T1
            T1 ← 2T1
        else:
            T1 ← T0 + T1
            T0 ← 2T0
    return T0
```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$\begin{aligned} T_0 &= P \cdot 2 &&= 2P \\ T_1 &= P \cdot 2 + P + 2P &&= 5P \end{aligned}$$

The Montgomery ladder

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return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2^2 = 4P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

```
function scalar-mult(k, P):
    T_0 ← O
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```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10011)_2$

$$T_0 = P \cdot 2^2 = 4P$$
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    else:
      $T_1 \leftarrow T_0 + T_1$
      $T_0 \leftarrow 2T_0$
  return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (100\underline{1}1)_2$

$$T_0 = P \cdot 2^2 + 5P = 9P$$
$$T_1 = P \cdot 2 + P + 2P = 5P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

function scalar-mult(k, P)**:**
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            $T_0 \leftarrow 2T_0$
    return T_0

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (10\underline{0}11)_2$

$$T_0 = P \cdot 2^2 + 5P = 9P$$
$$T_1 = (P \cdot 2 + P + 2P) \cdot 2 = 10P$$

The Montgomery ladder

- Algorithm proposed by Montgomery in 1987:

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- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1001\underline{1})_2$

$$T_0 = P \cdot 2^2 + 5P = 9P$$

$$T_1 = (P \cdot 2 + P + 2P) \cdot 2 = 10P$$

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- Properties:
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$$T_0 = P \cdot 2^2 + 5P + 10P = 19P$$
$$T_1 = (P \cdot 2 + P + 2P) \cdot 2 = 10P$$

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```

- Properties:
  - perform one addition and one doubling at each step
  - ensure that both results are used in the next step
  - loop invariant: $T_1 = T_0 + P$
- Example: $k = 19 = (1001\underline{1})_2$

$$T_0 = P \cdot 2^2 + 5P + 10P = 19P$$
$$T_1 = (P \cdot 2 + P + 2P) \cdot 2^2 = 20P$$

The Montgomery ladder

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- Example: $k = 19 = (10011)_2$

$$T_0 = P \cdot 2^2 + 5P + 10P = 19P$$
$$T_1 = (P \cdot 2 + P + 2P) \cdot 2^2 = 20P$$

More security issues

function scalar-mult(k, P)**:**
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    return T_0
```

- The conditional branches depend on the value of secret bit k_i

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return T_0

- The conditional branches depend on the value of secret bit k_i
  $\Rightarrow$ might be vulnerable to branch prediction attacks

More security issues

function scalar-mult(k, P)**:**

$T_0 \leftarrow \mathcal{O}$

$T_1 \leftarrow P$

for $i \leftarrow n - 1$ **downto** 0**:**

$T_{1-k_i} \leftarrow T_0 + T_1$

$T_{k_i} \leftarrow 2T_{k_i}$

return T_0

- The conditional branches depend on the value of secret bit k_i
  $\Rightarrow$ might be vulnerable to branch prediction attacks

- Compute indices for T_0 and T_1 from k_i?

More security issues

function scalar-mult(k, P)**:**
  $T_0 \leftarrow \mathcal{O}$
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  for $i \leftarrow n-1$ **downto** 0**:**
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  return T_0

- The conditional branches depend on the value of secret bit k_i
  $\Rightarrow$ might be vulnerable to branch prediction attacks
- Compute indices for T_0 and T_1 from k_i?
  - memory accesses to T_0 or T_1 depend on secret bit k_i

More security issues

function scalar-mult(k, P)**:**

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return T_0

- The conditional branches depend on the value of secret bit k_i
  $\Rightarrow$ might be vulnerable to branch prediction attacks
- Compute indices for T_0 and T_1 from k_i?
  - memory accesses to T_0 or T_1 depend on secret bit k_i

  $\Rightarrow$ might be vulnerable to cache attacks

More security issues

function scalar-mult(k, P):

$$
\begin{aligned}
& T_0 \leftarrow \mathcal{O} \\
& T_1 \leftarrow P \\
& \textbf{for } i \leftarrow n-1 \textbf{ downto } 0\textbf{:} \\
& \quad M \leftarrow (k_i \dots k_i)_2 \\
& \quad R \leftarrow T_0 + T_1 \\
& \quad S \leftarrow 2((\overline{M} \& T_0) \mid (M \& T_1)) \\
& \quad T_0 \leftarrow (\overline{M} \& S) \mid (M \& R) \\
& \quad T_1 \leftarrow (\overline{M} \& R) \mid (M \& S) \\
& \textbf{return } T_0
\end{aligned}
$$

- The conditional branches depend on the value of secret bit k_i
  $\Rightarrow$ might be vulnerable to branch prediction attacks
- Compute indices for T_0 and T_1 from k_i?
  - memory accesses to T_0 or T_1 depend on secret bit k_i

  $\Rightarrow$ might be vulnerable to cache attacks
- Use bit masking to avoid secret-dependent memory access patterns

Outline

I. Scalar multiplication

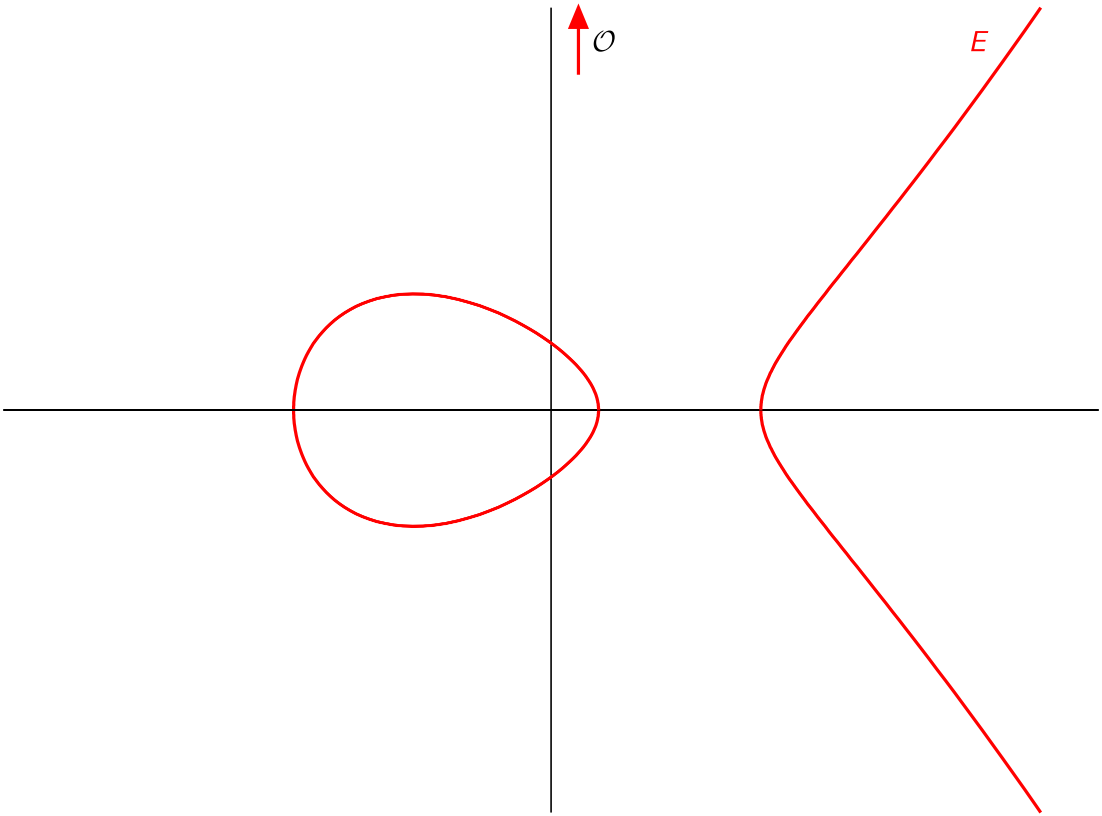
II. Elliptic curve arithmetic

III. Finite field arithmetic

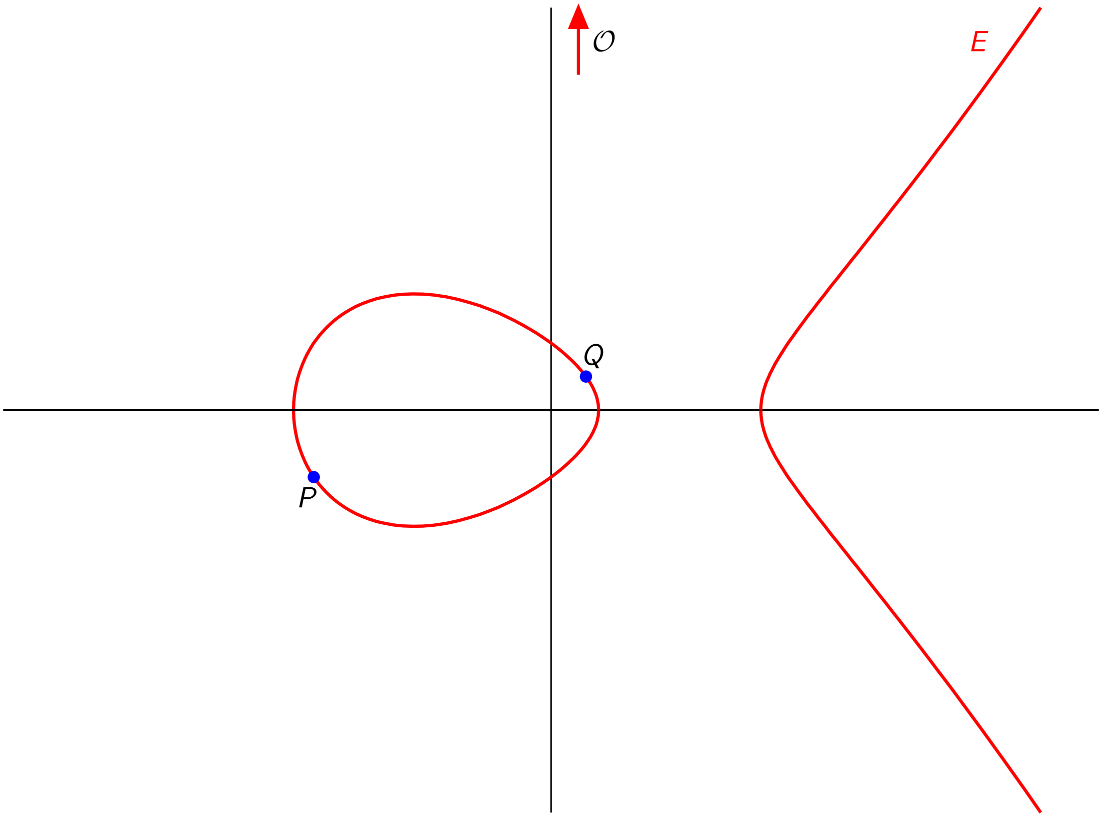
IV. Software considerations

V. Notions of hardware design

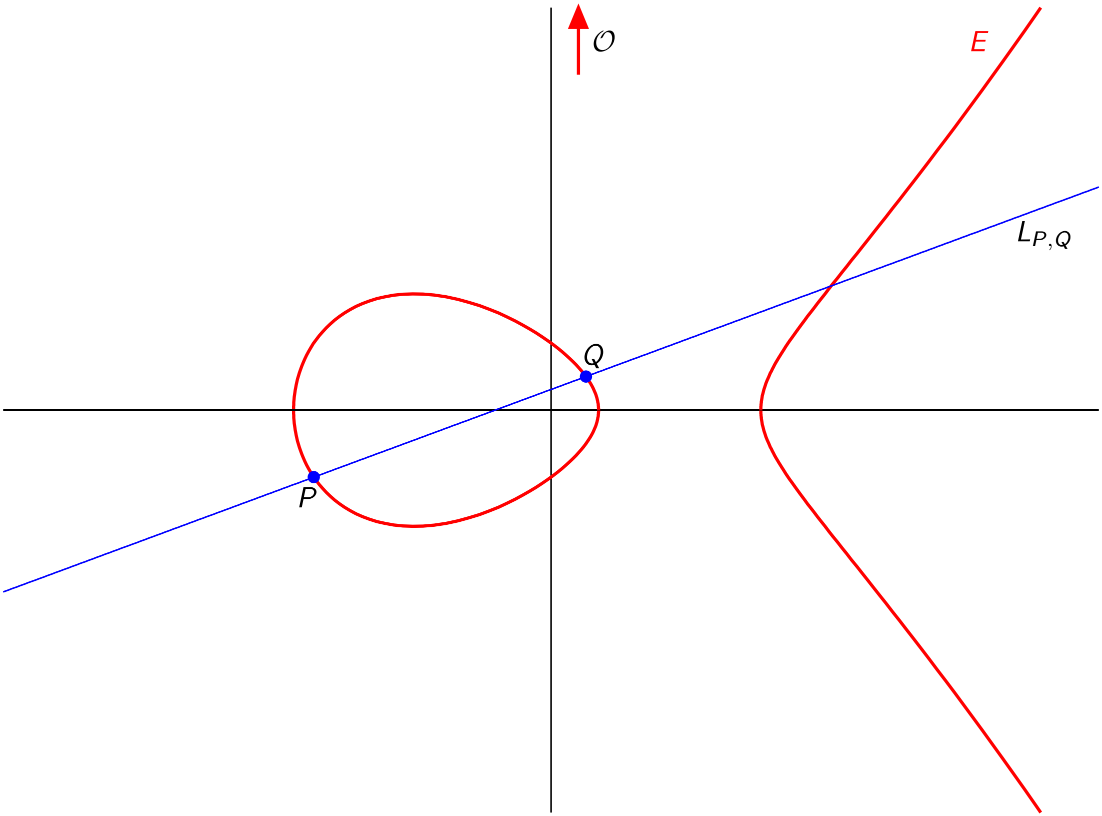
Addition and doubling



Addition and doubling



Addition and doubling



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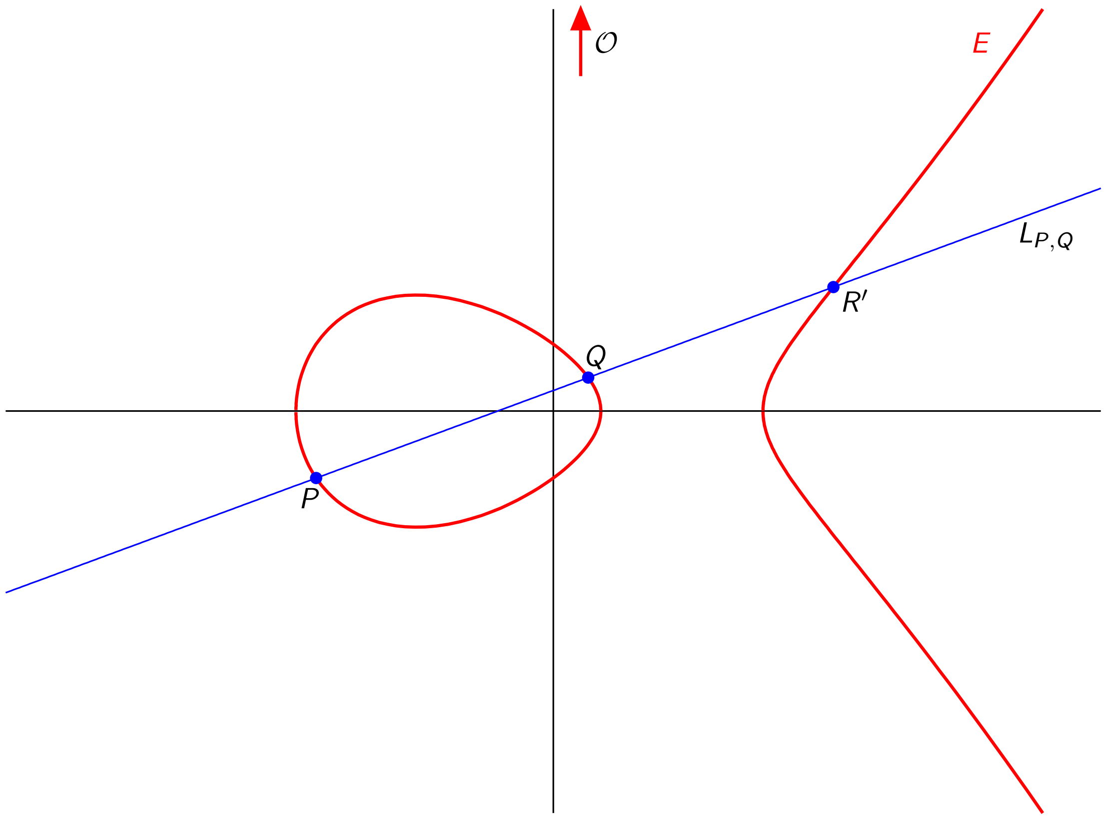

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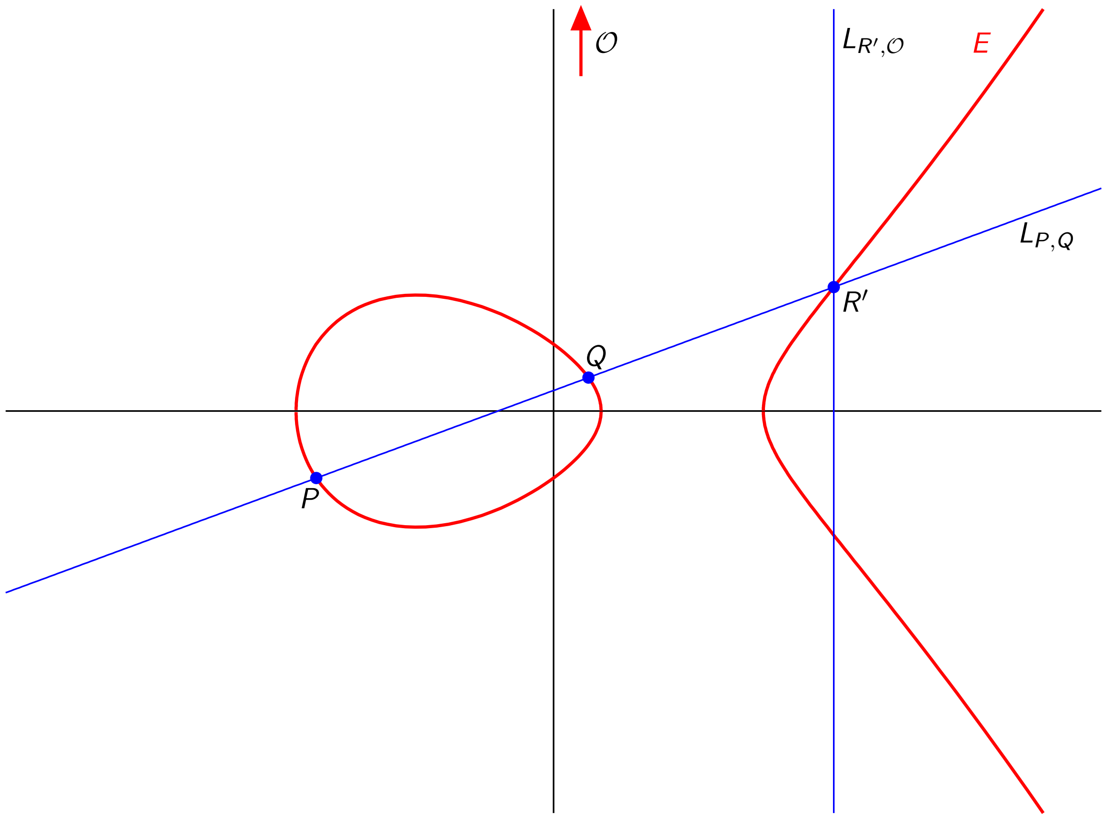

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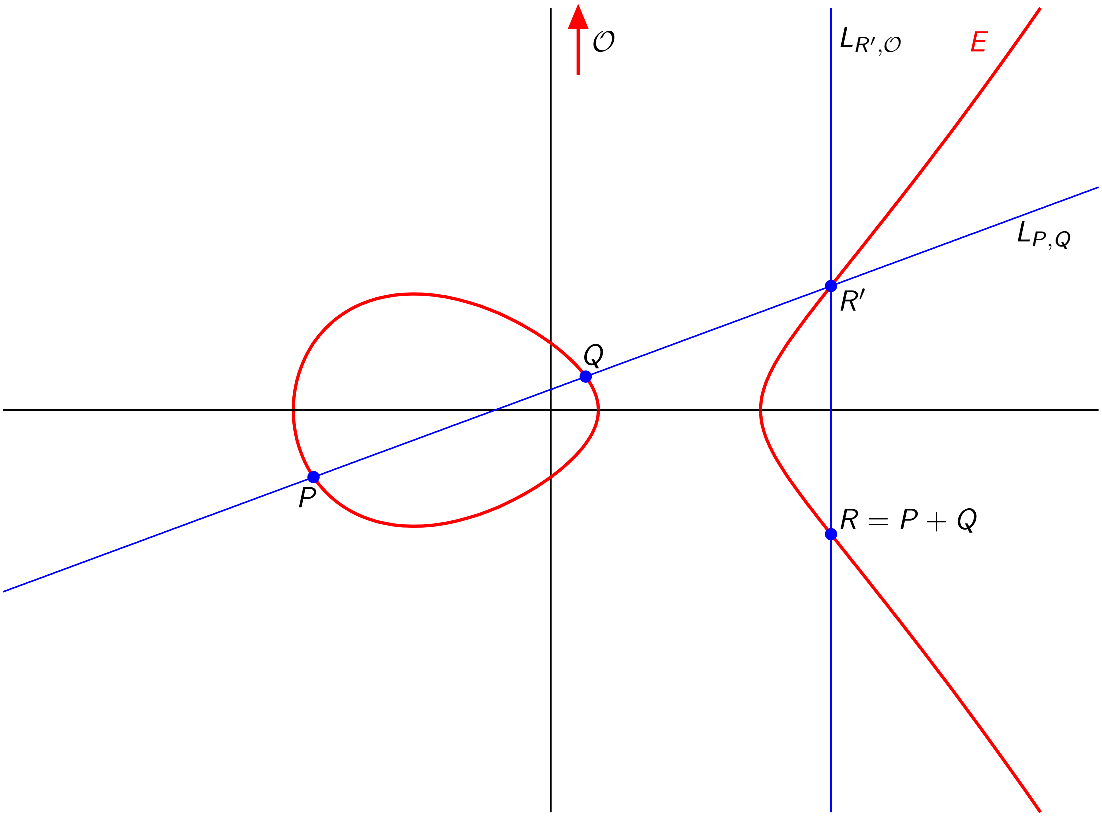
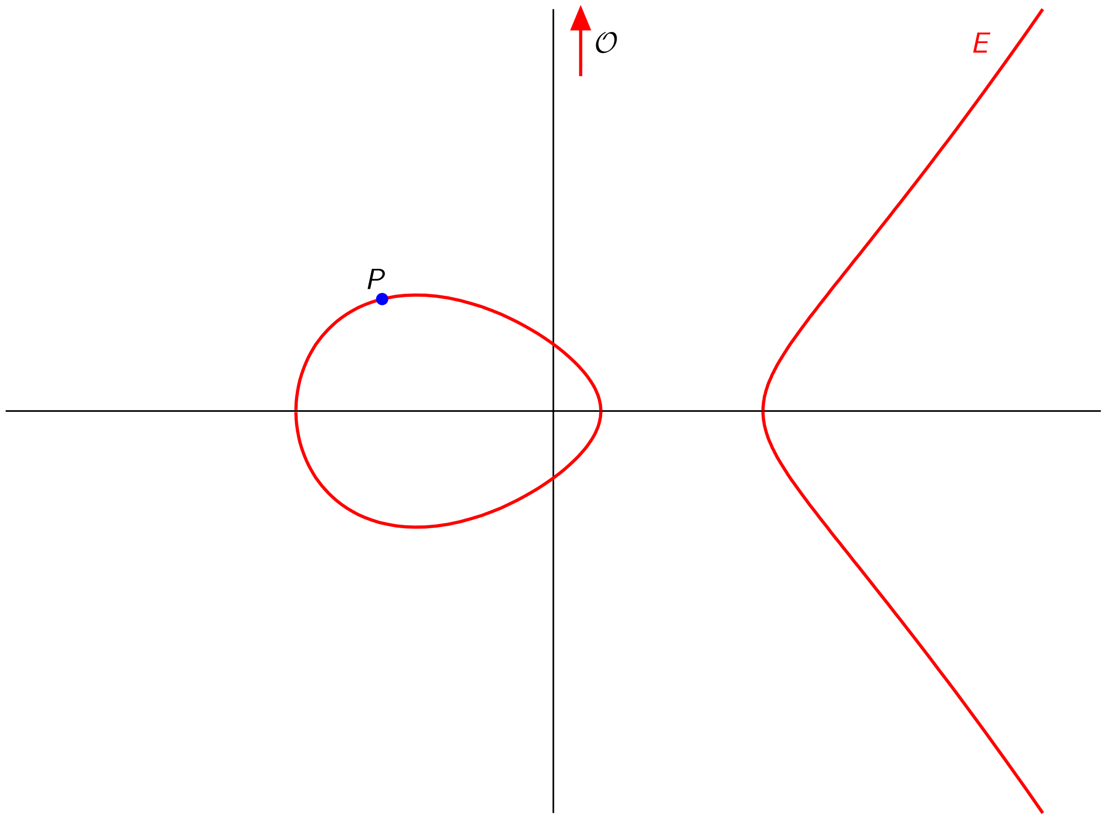
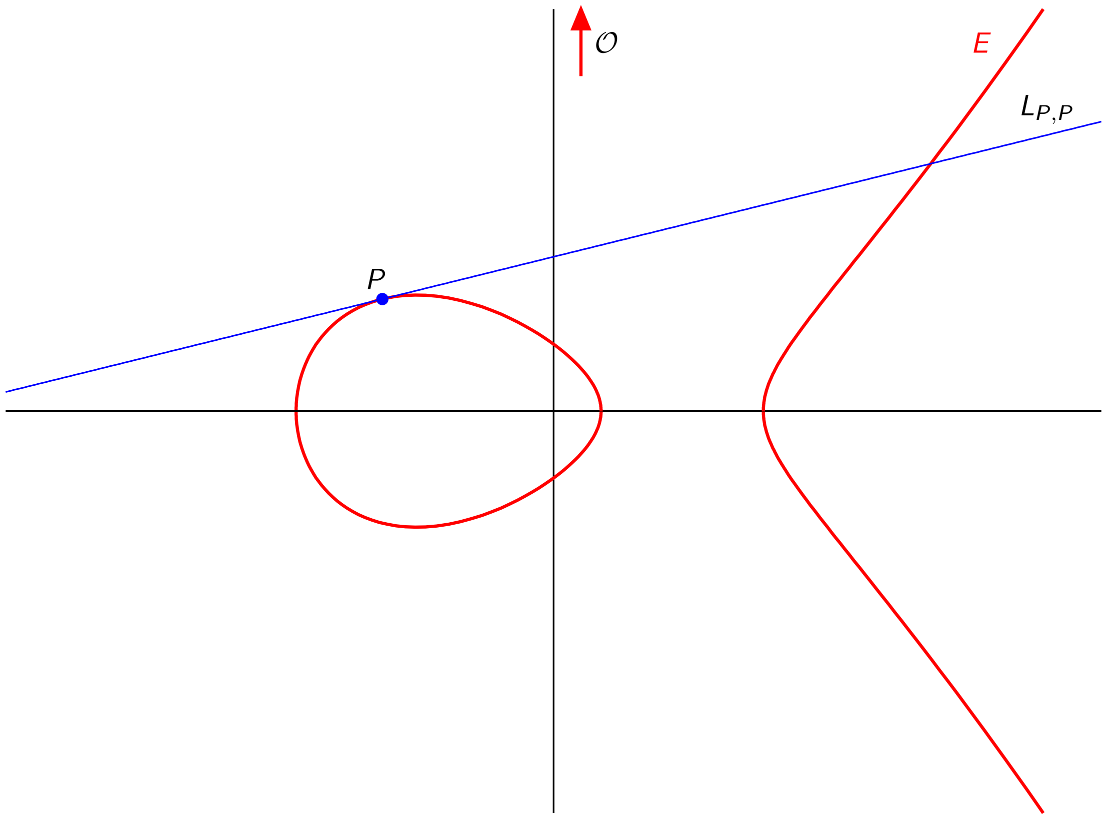


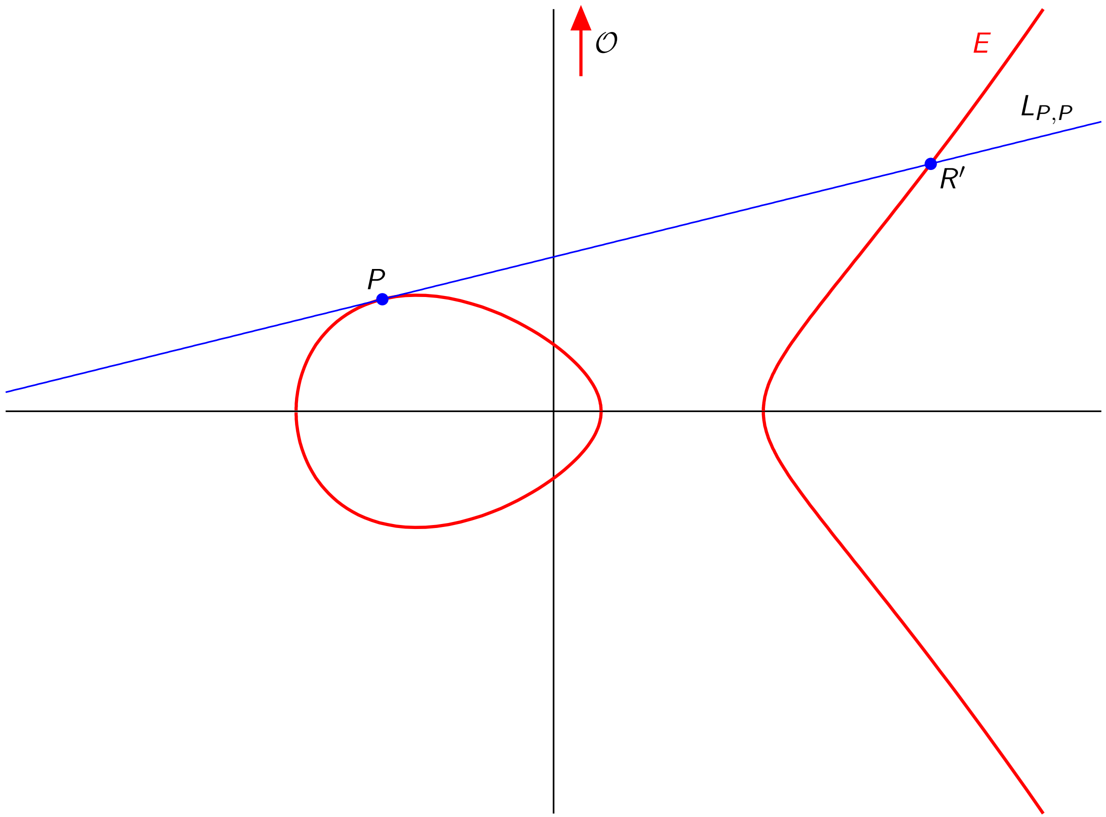
Addition and doubling



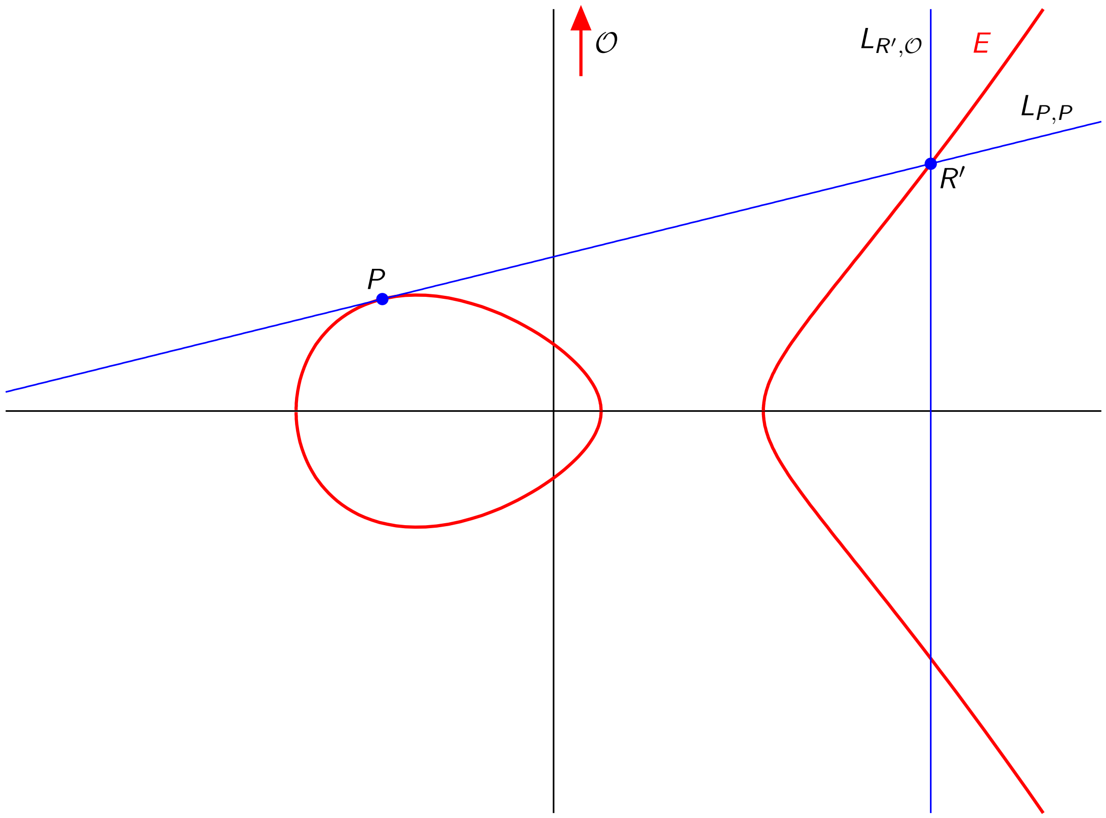
Addition and doubling



Addition and doubling



Addition and doubling



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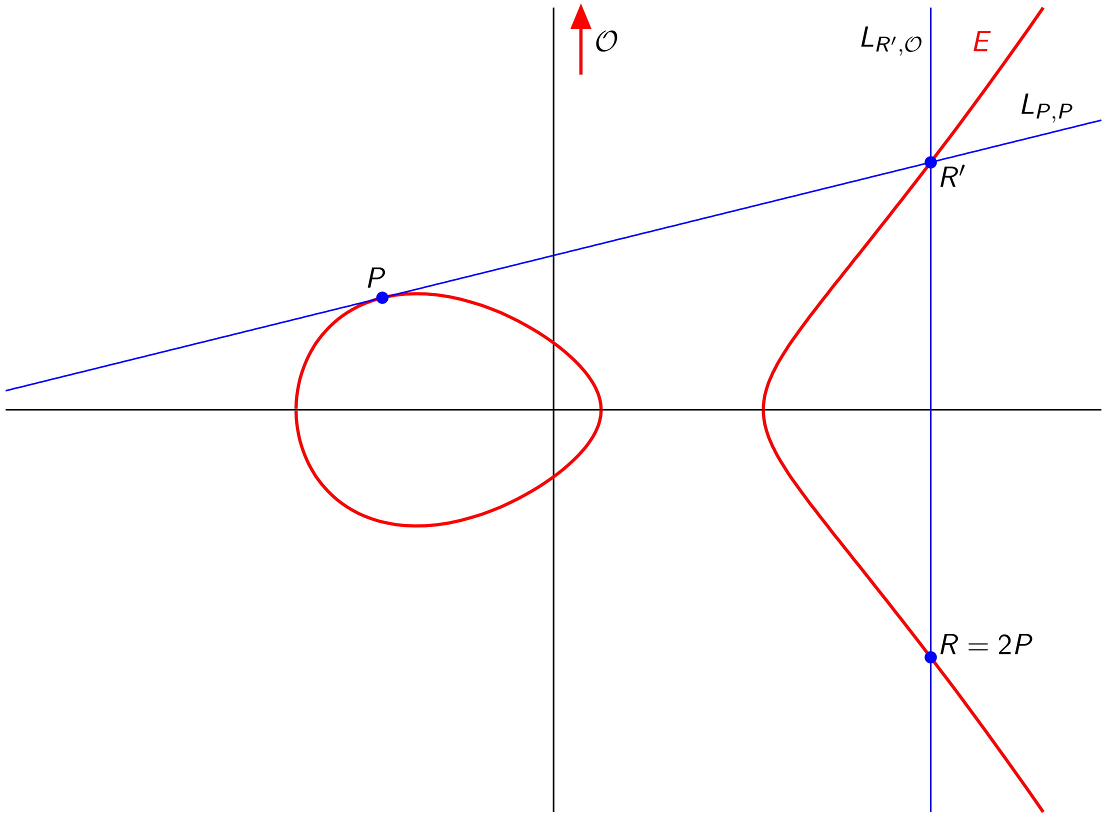

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- Explicit-Formula Database (by Bernstein and Lange):

`http://hyperelliptic.org/EFD/`

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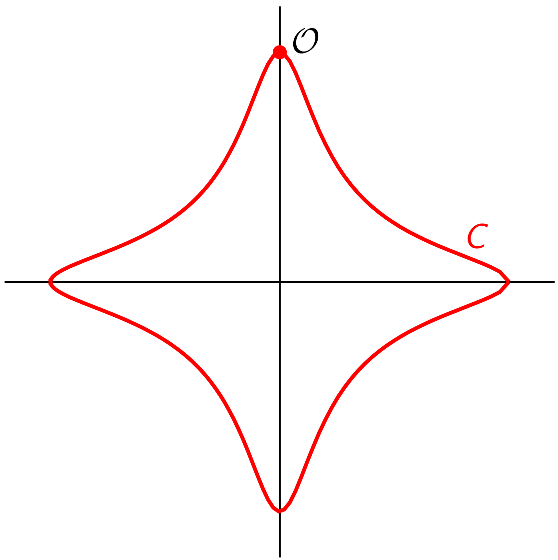

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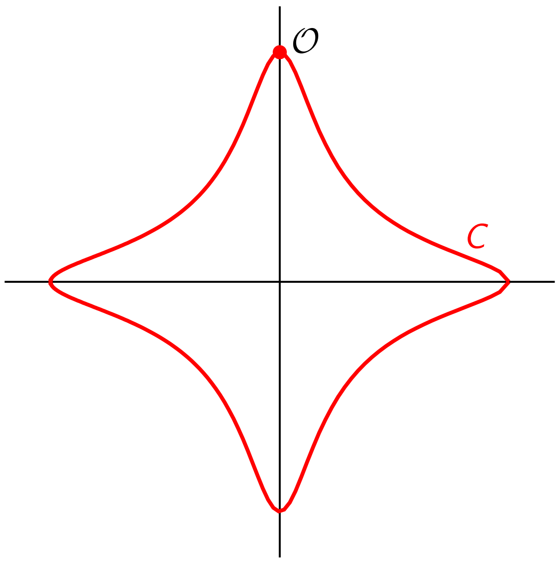

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  - doubling: same as addition
- Strongly unified and complete addition law:
  - works for both addition and doubling
  - no exceptional case

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- Generalization by Bernstein *et al.* (2008): twisted Edwards curves

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  - birationally equivalent to Montgomery curves

Outline

I. Scalar multiplication

II. Elliptic curve arithmetic

III. Finite field arithmetic

IV. Software considerations

V. Notions of hardware design

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  $\Rightarrow$ elements of $\mathbb{F}_q$ represented using several words

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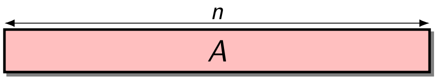

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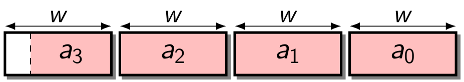

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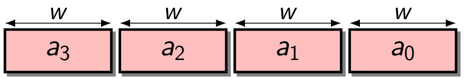

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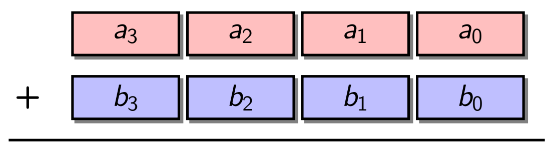

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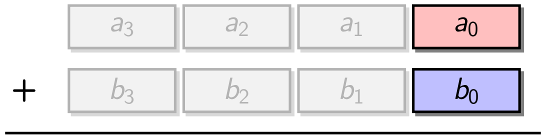

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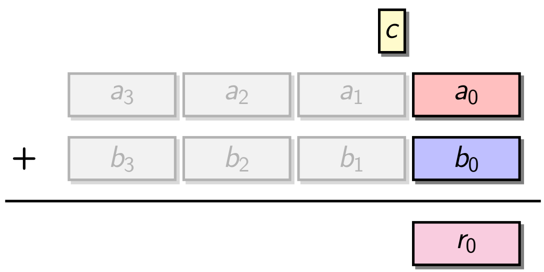

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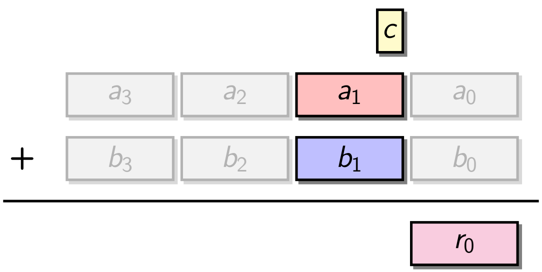

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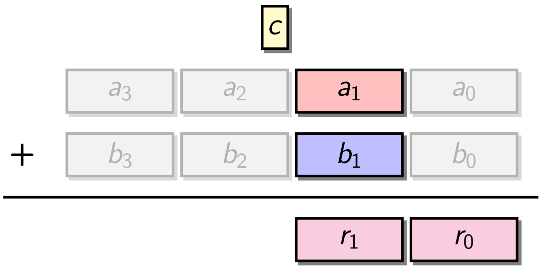

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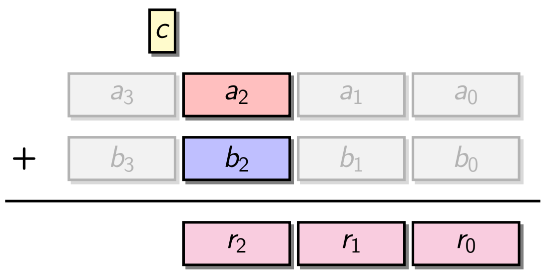

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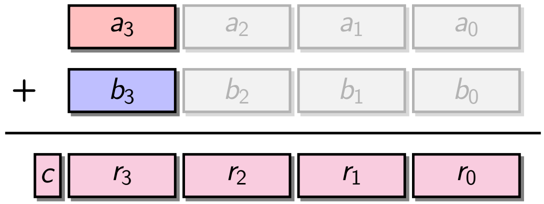

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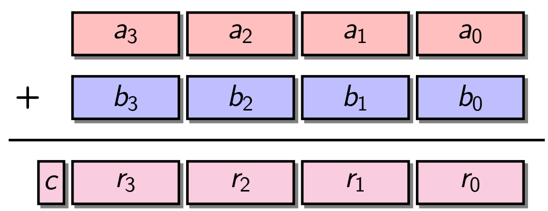

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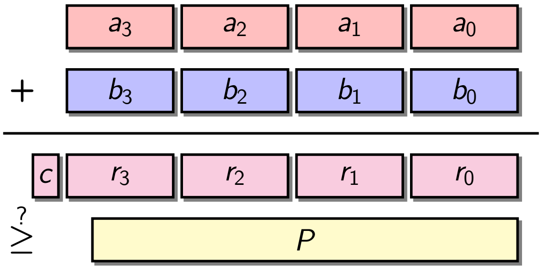

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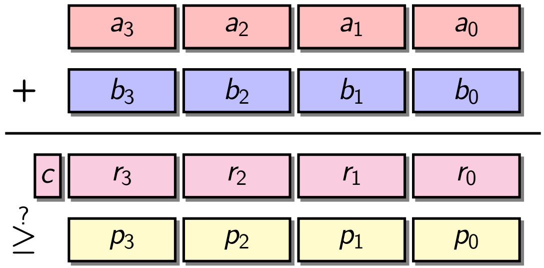

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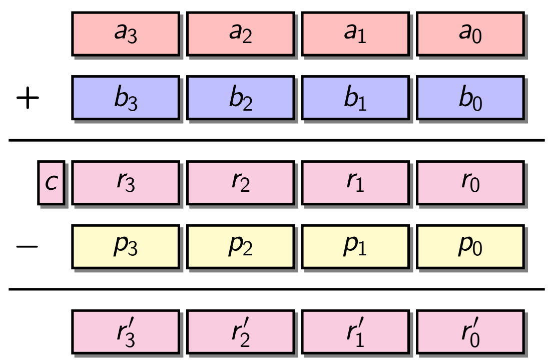

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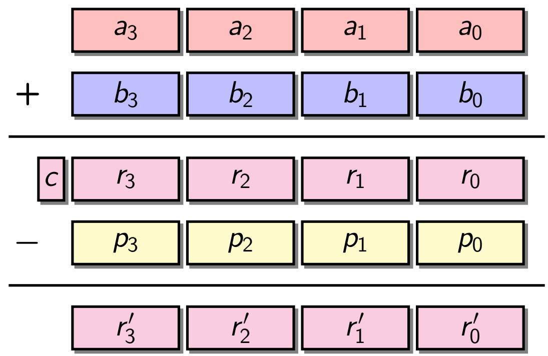

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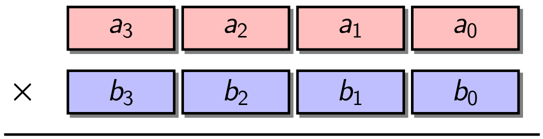

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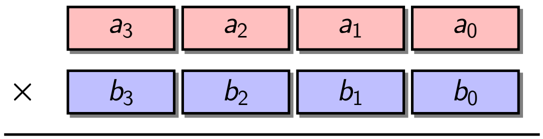

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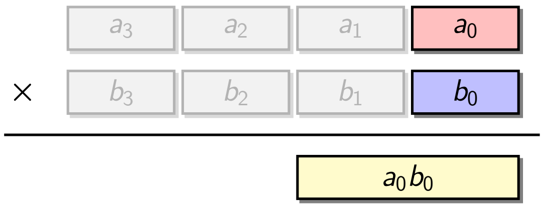

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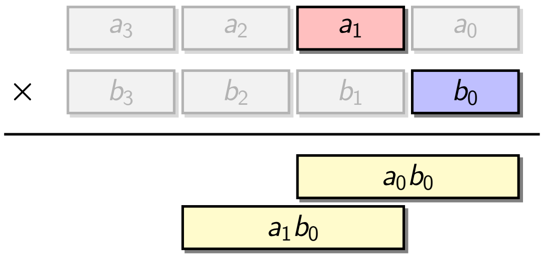

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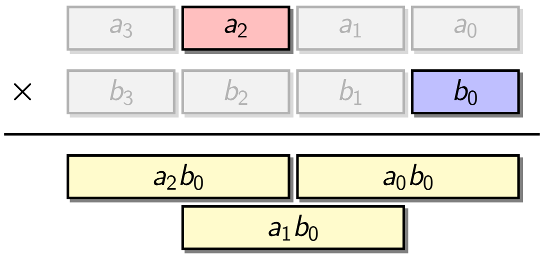

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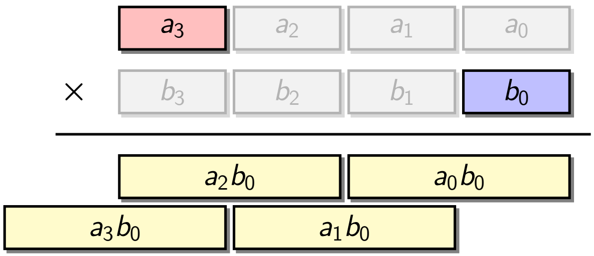

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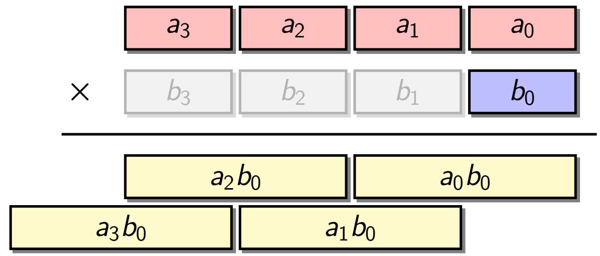

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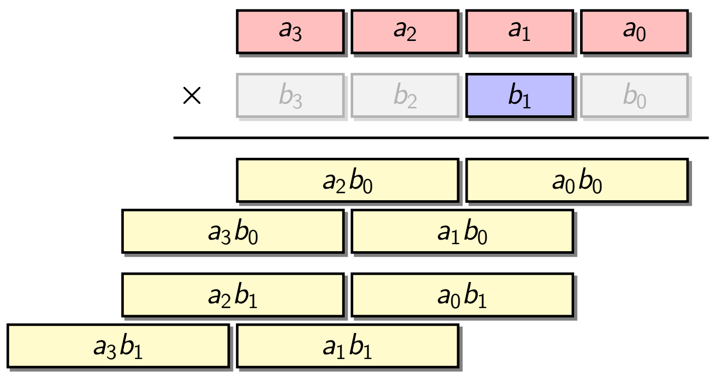

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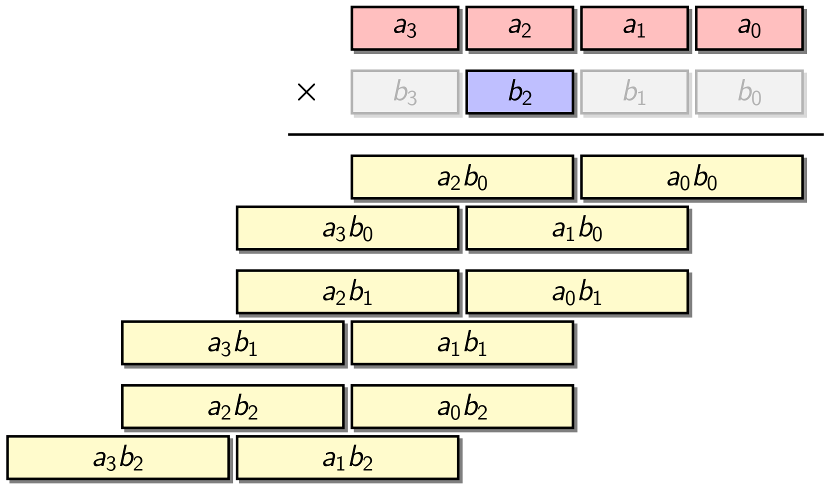

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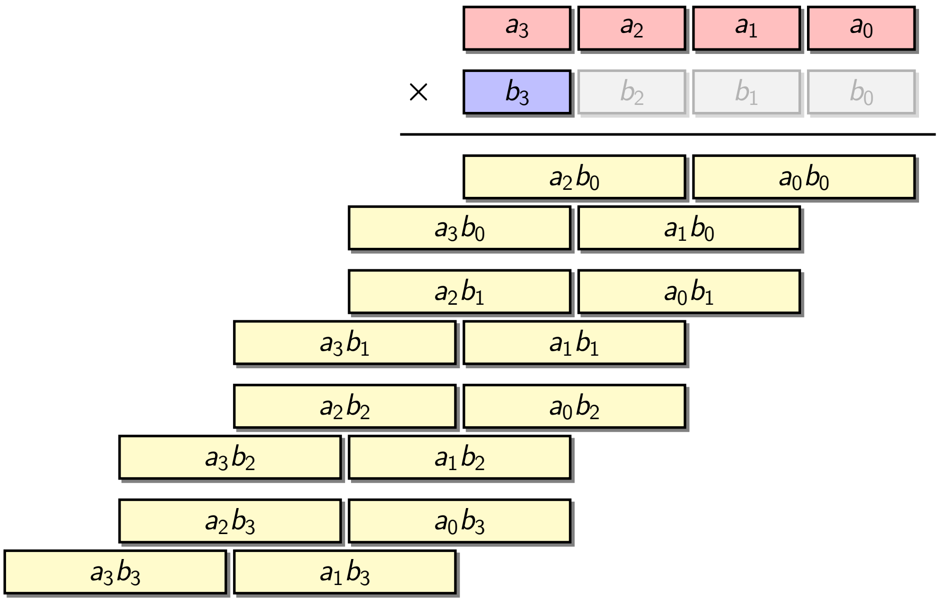

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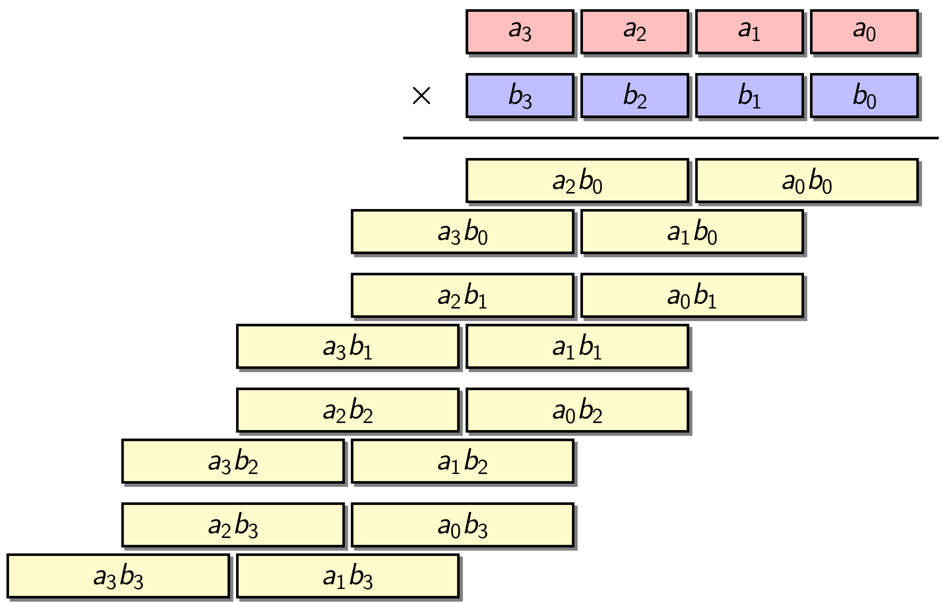

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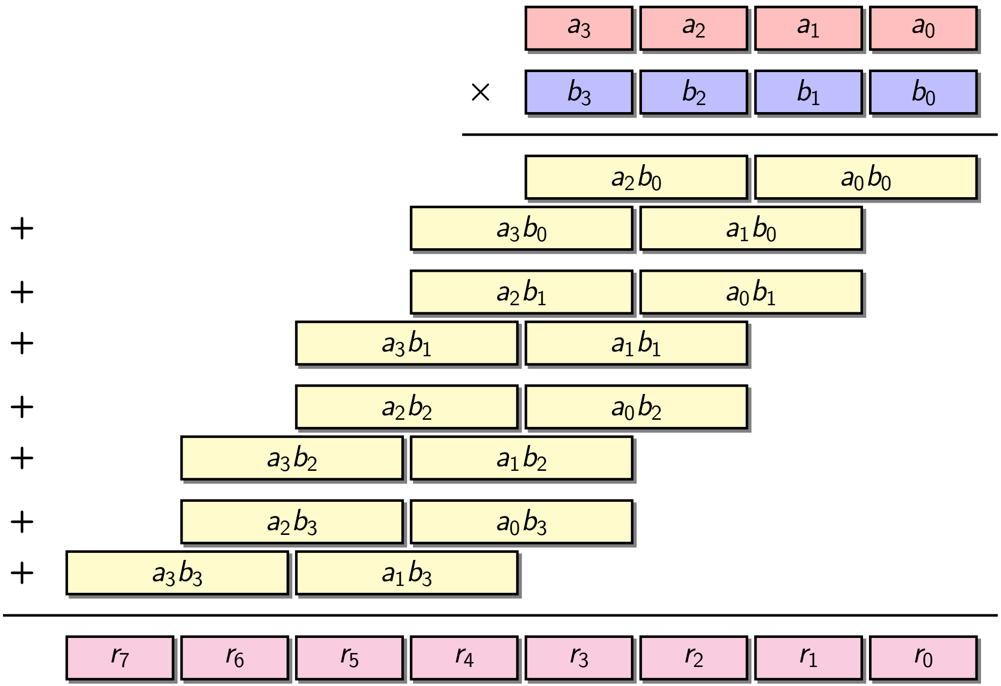

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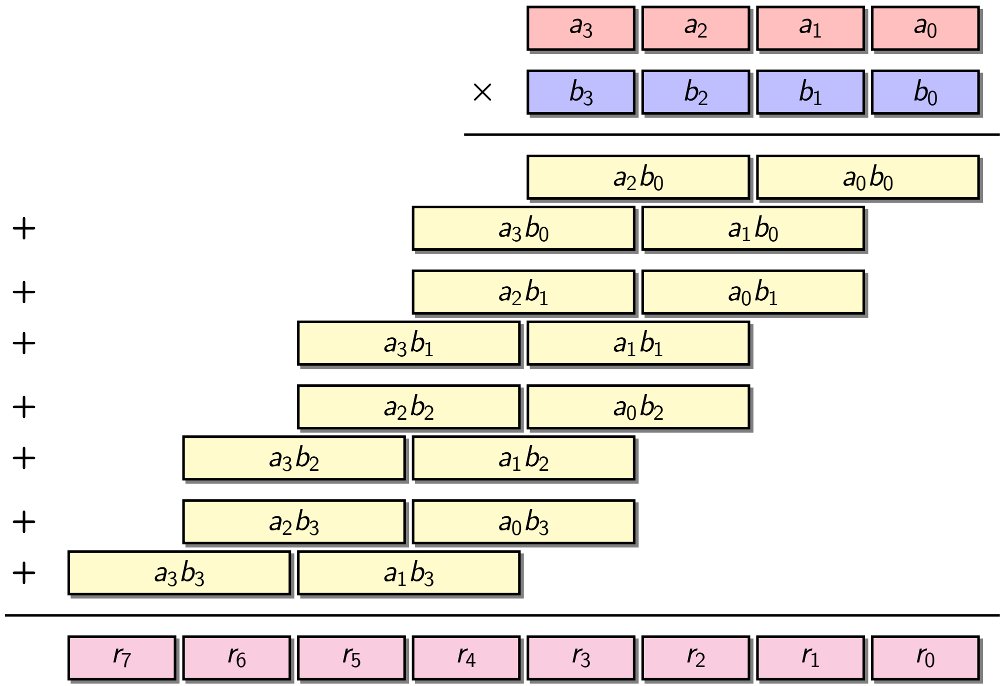

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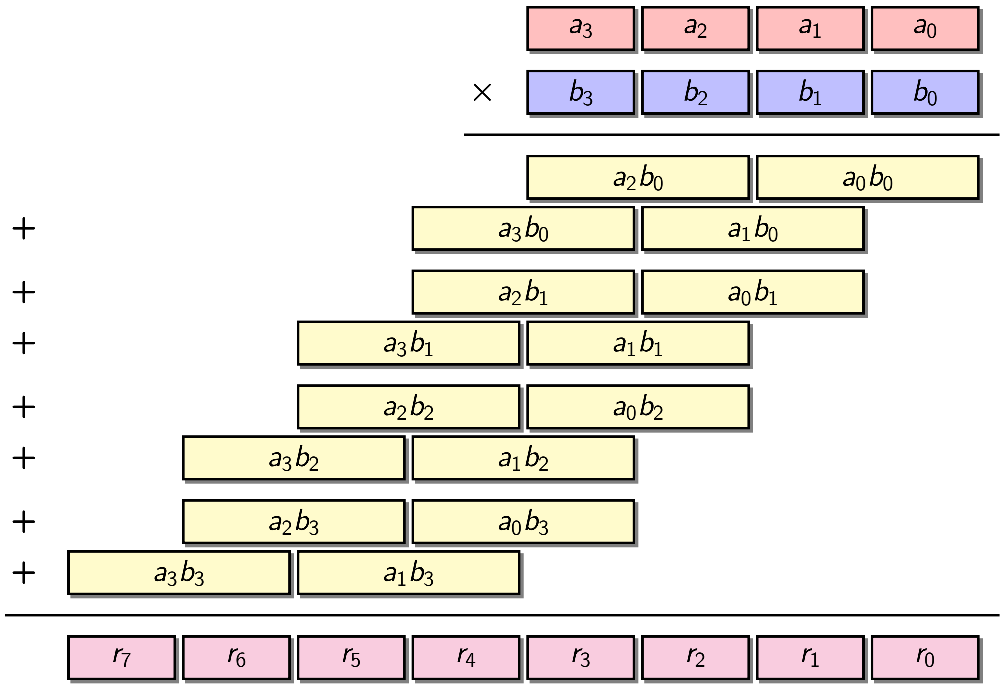

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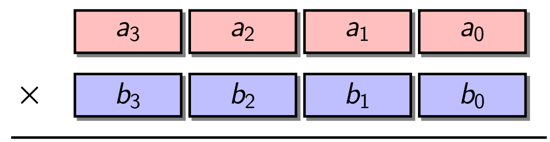

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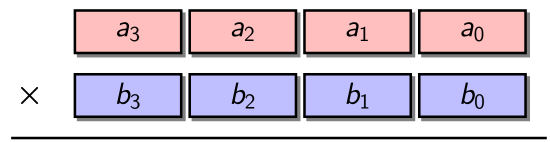

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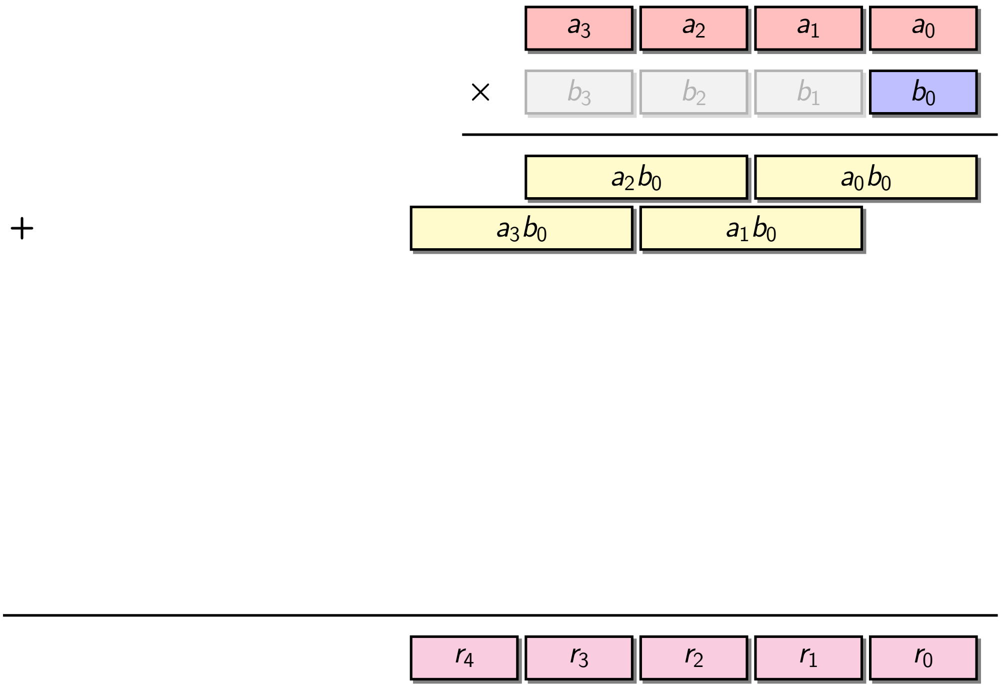

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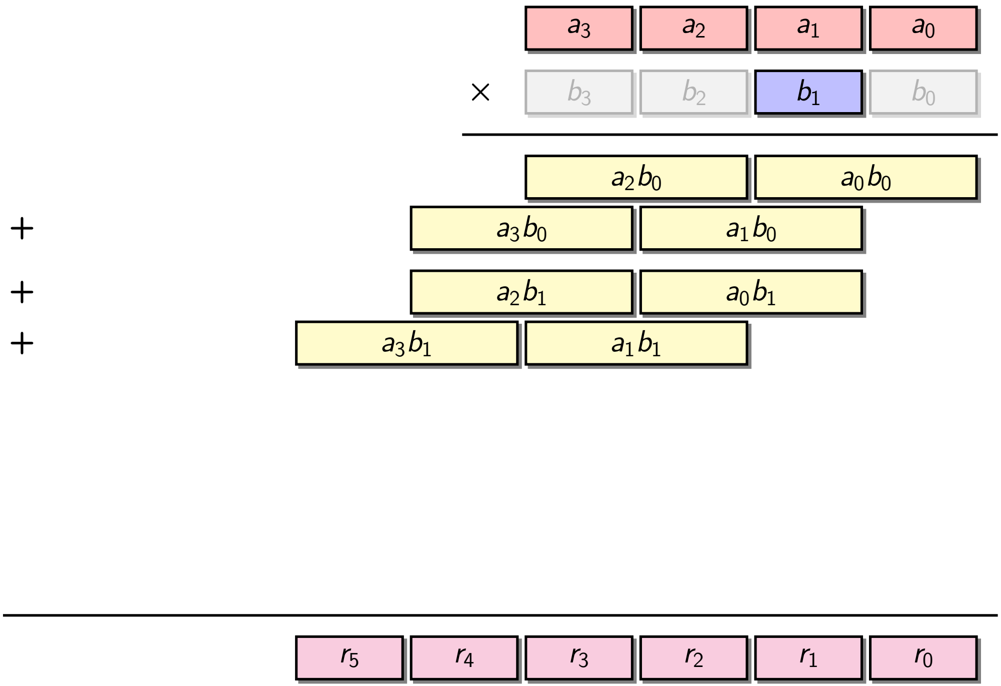

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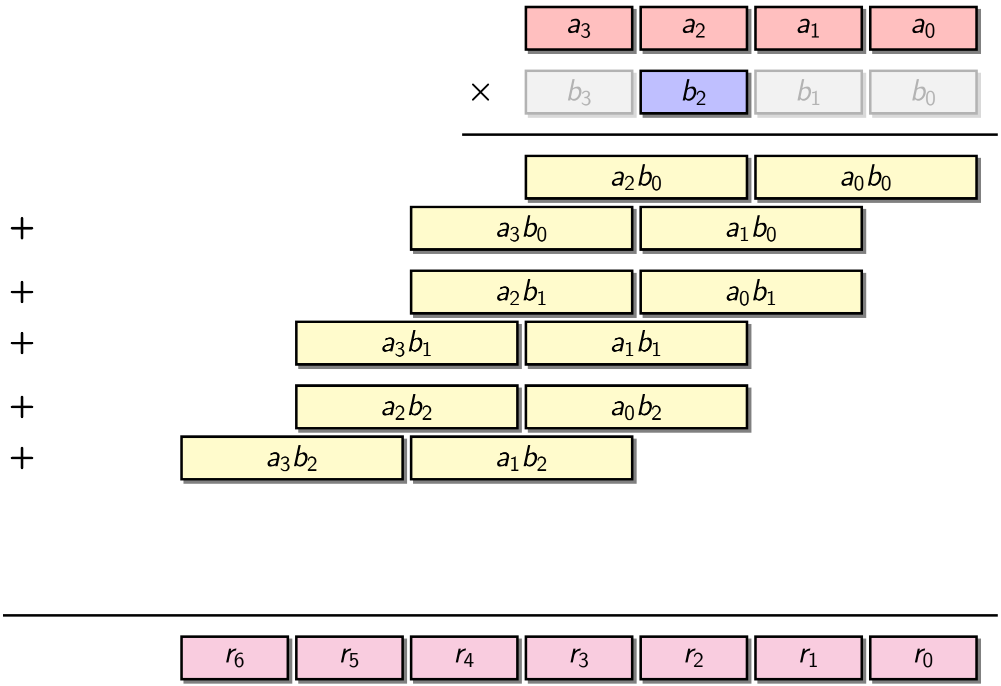

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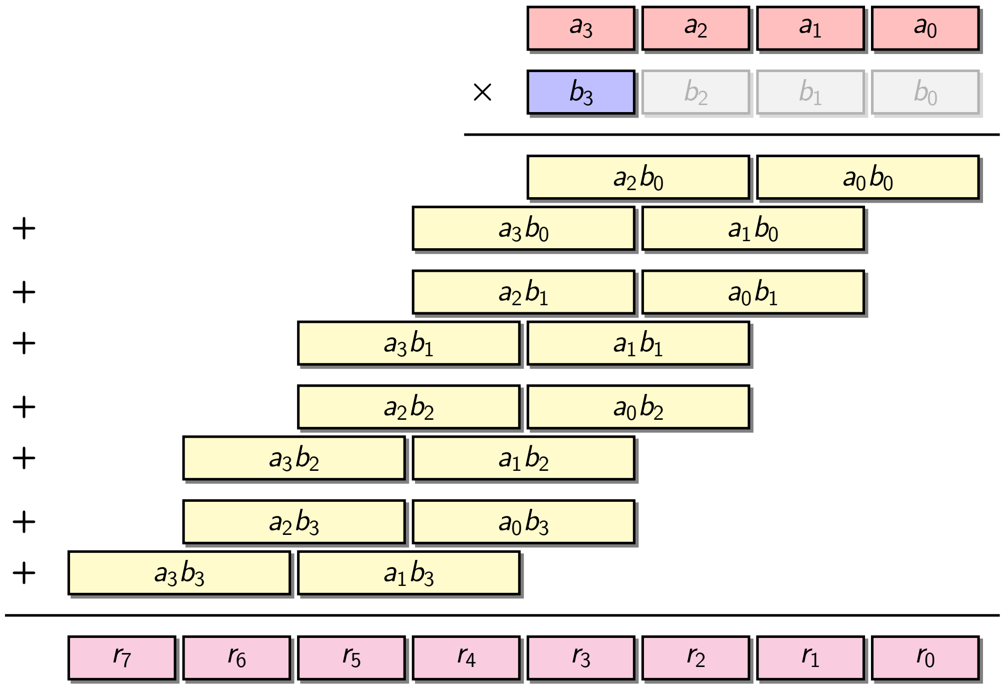

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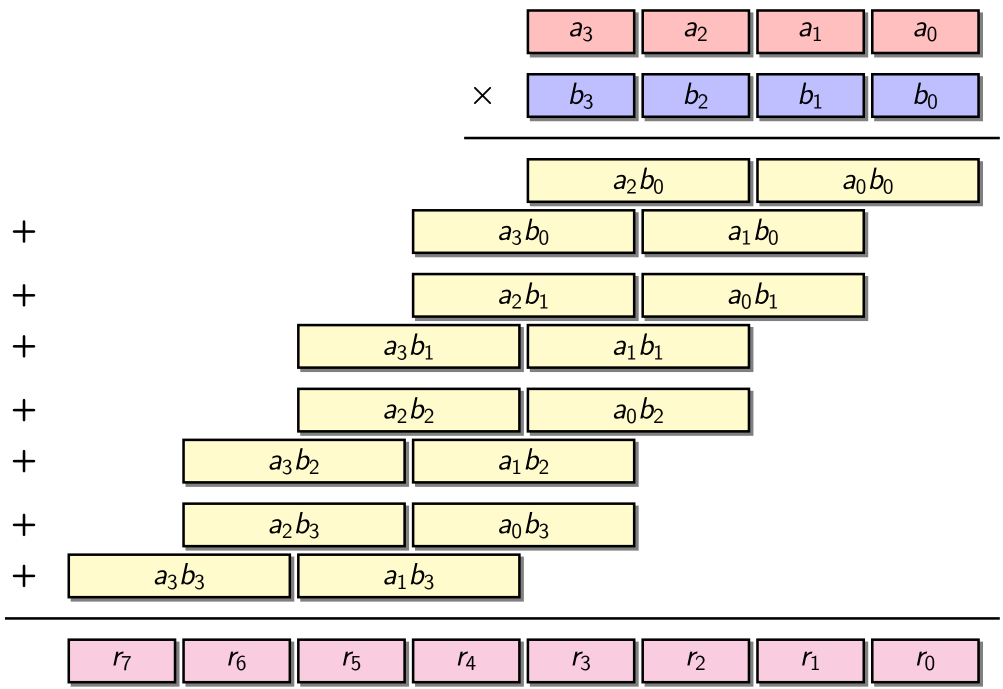

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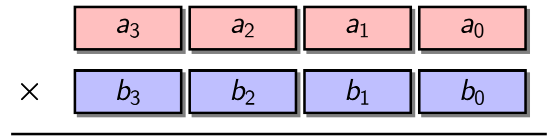

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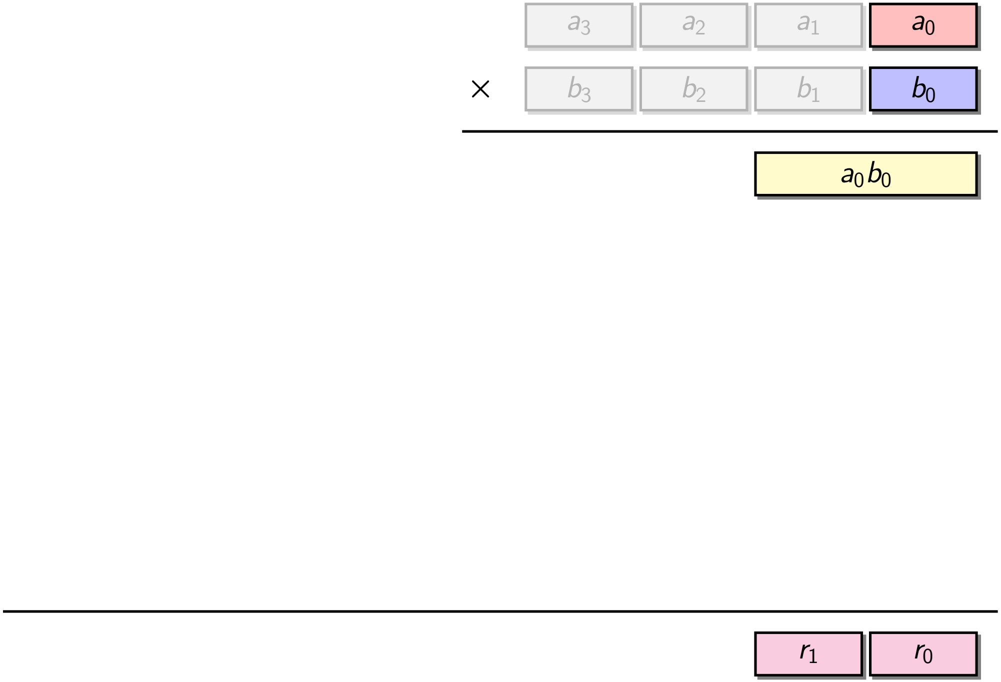

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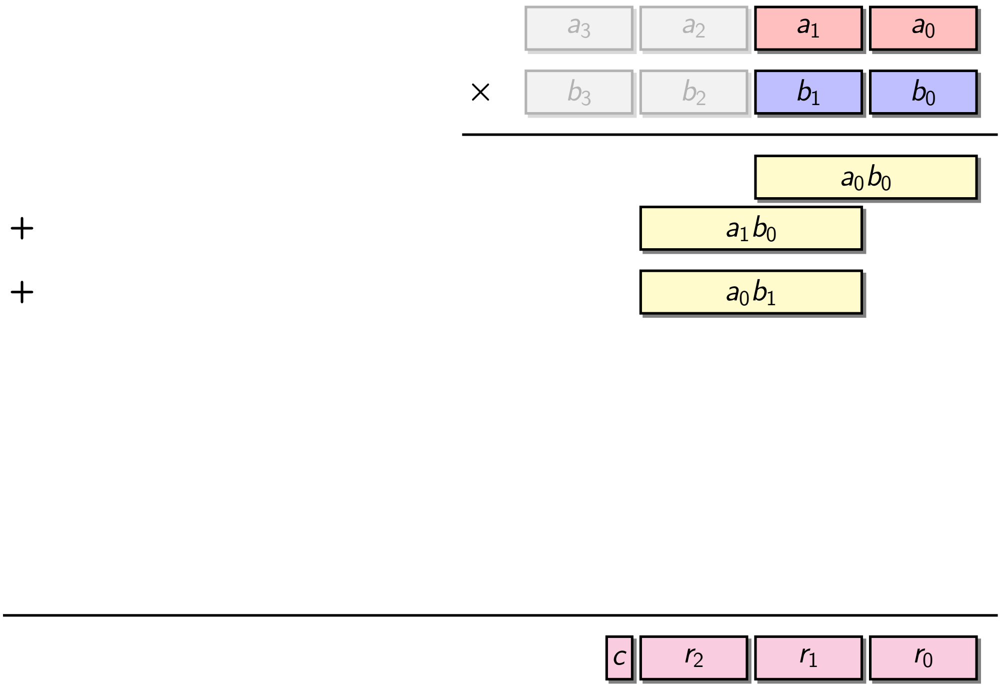

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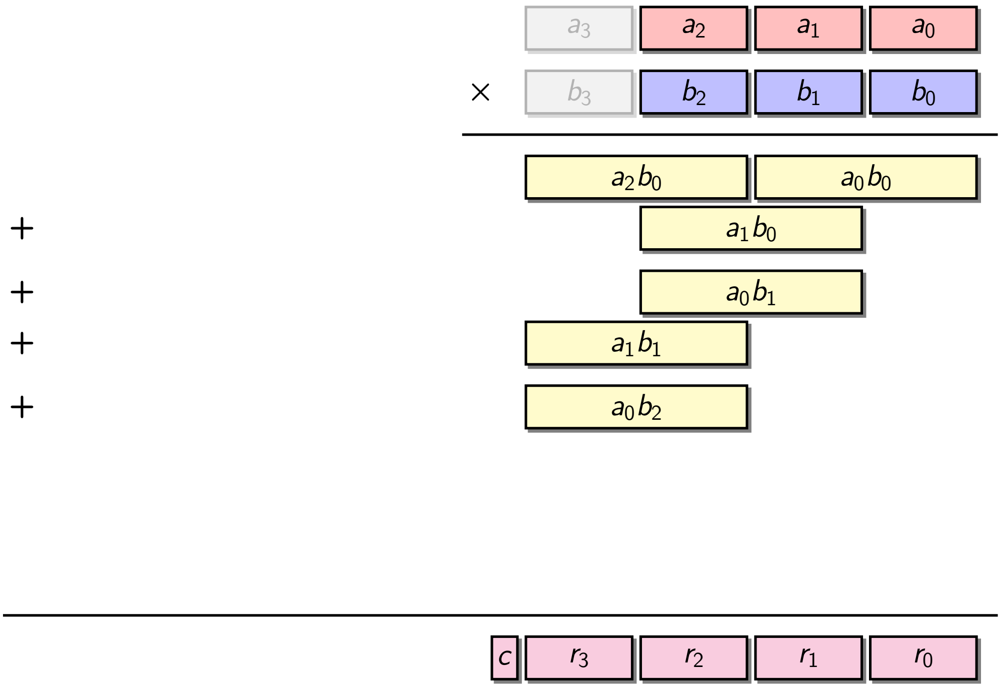

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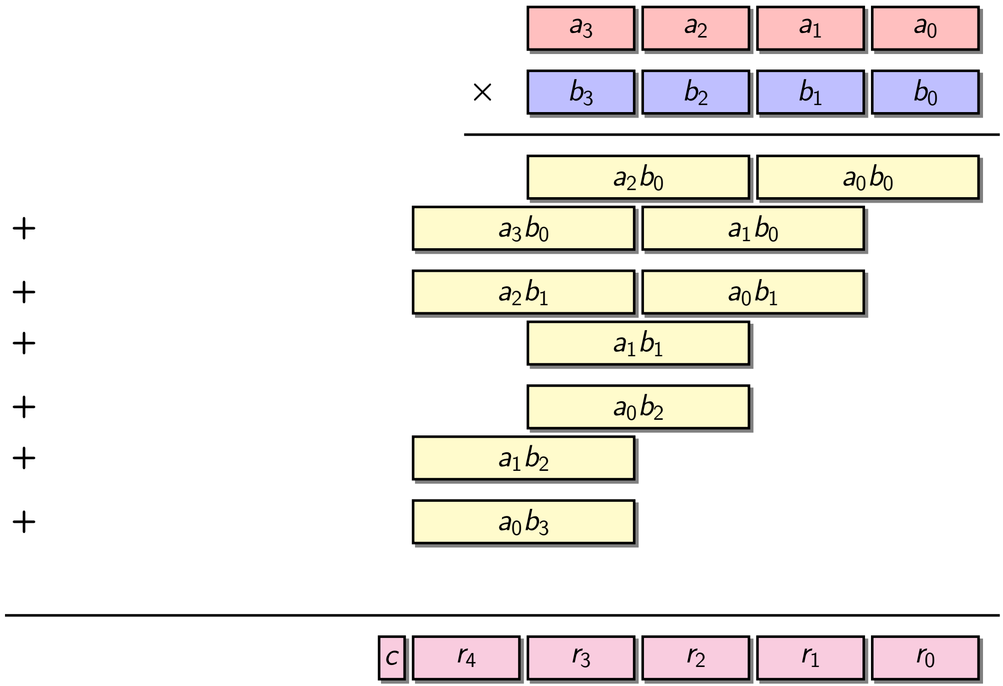

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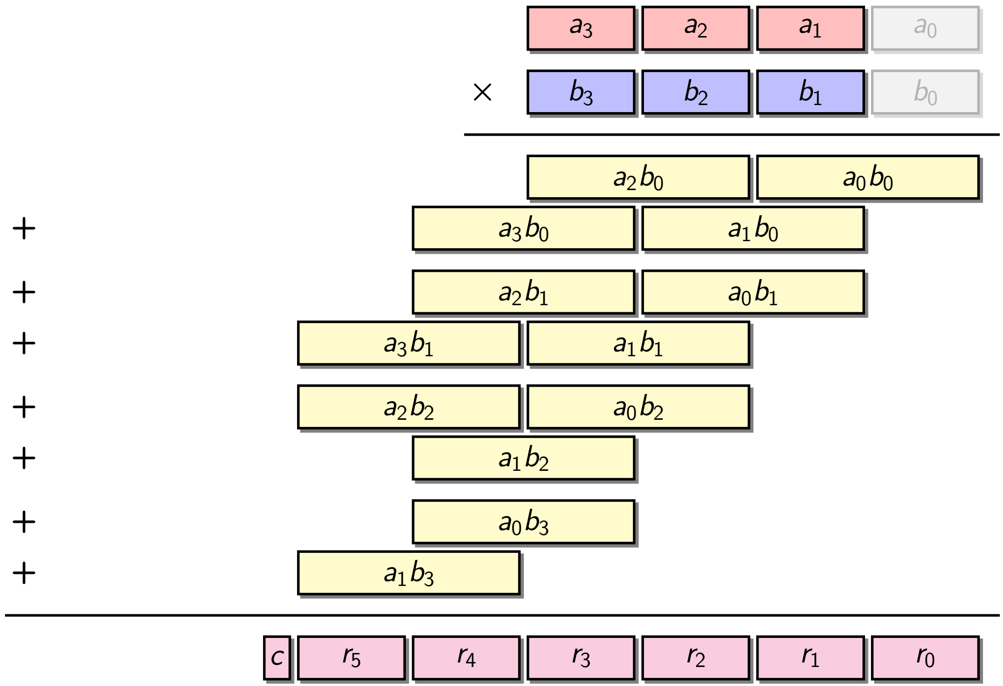

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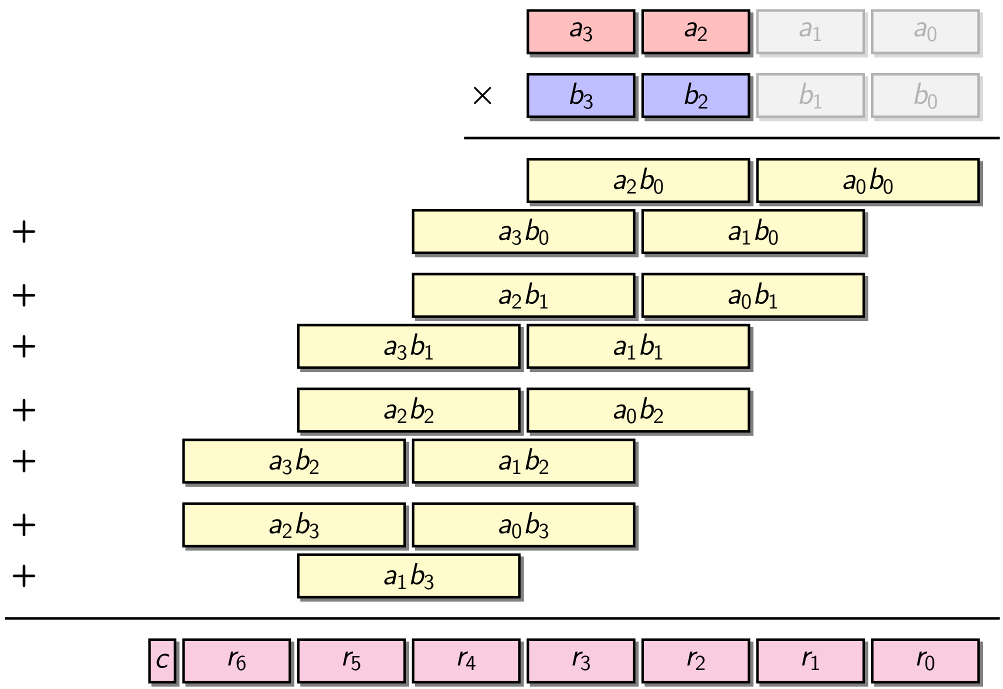

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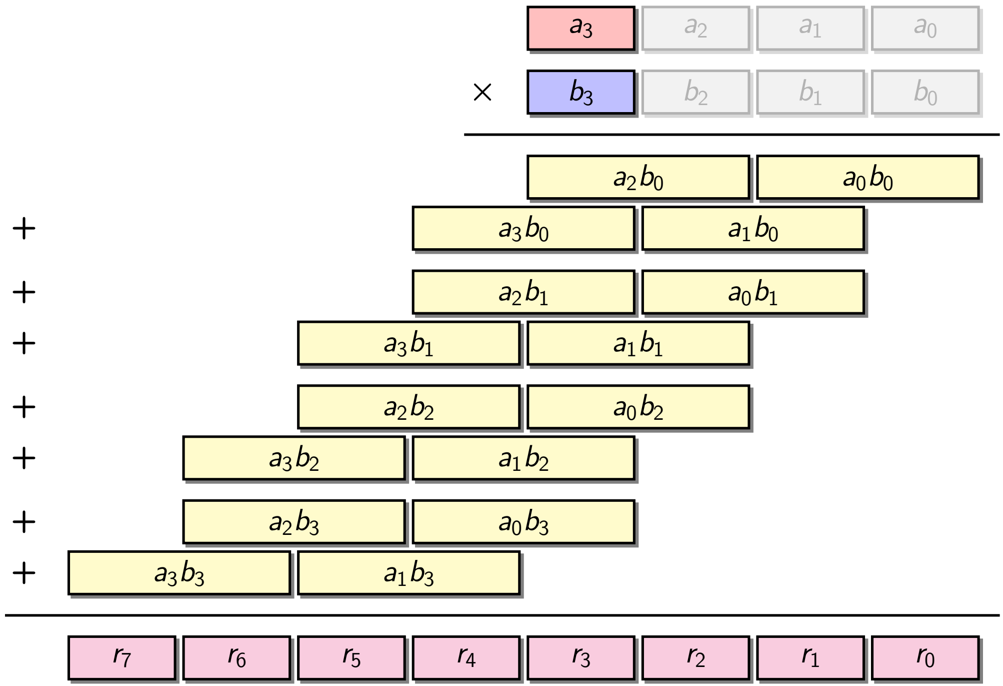

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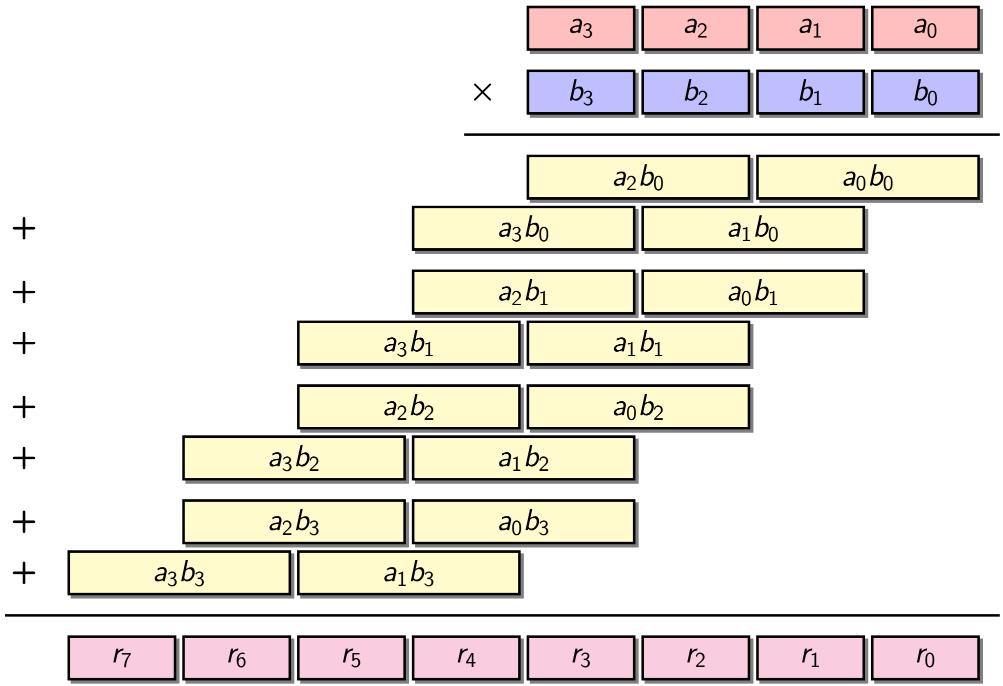

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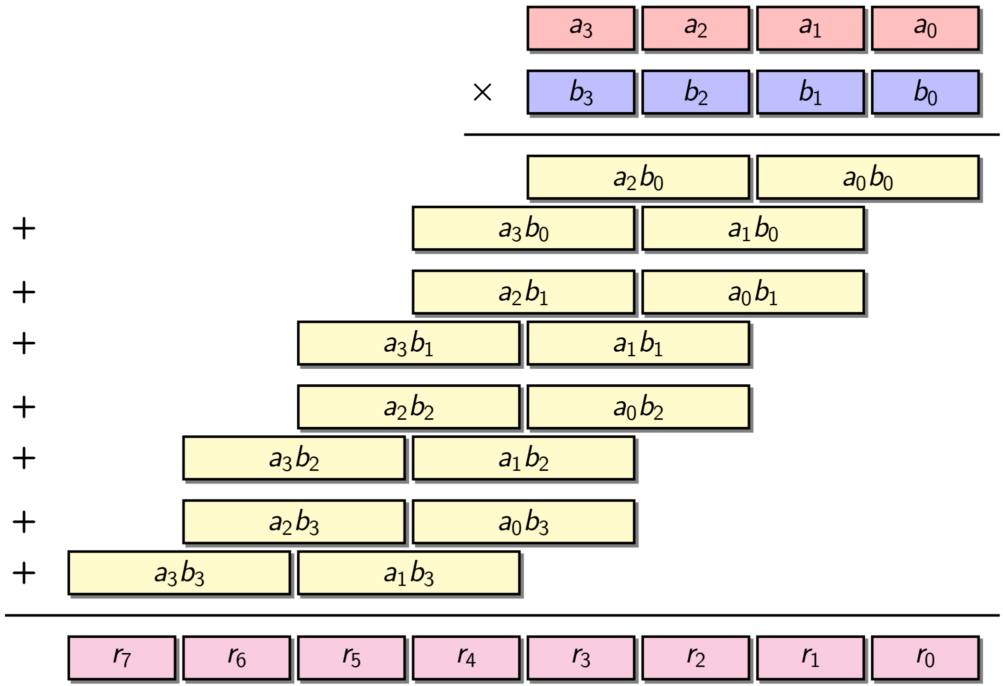

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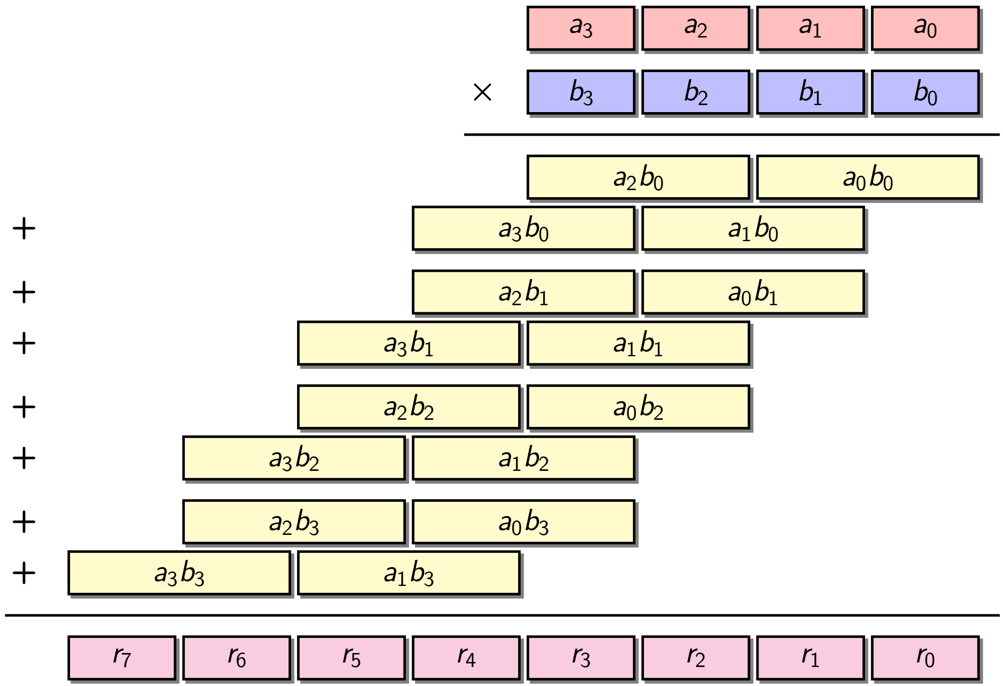

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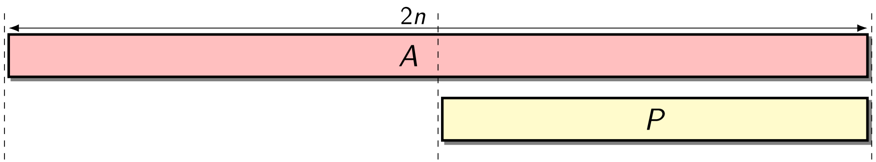

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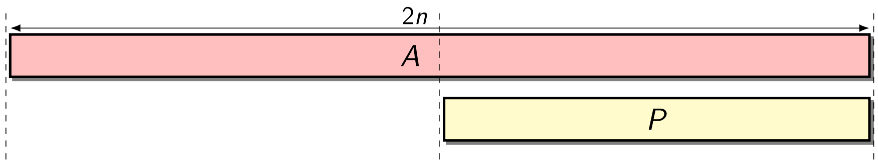

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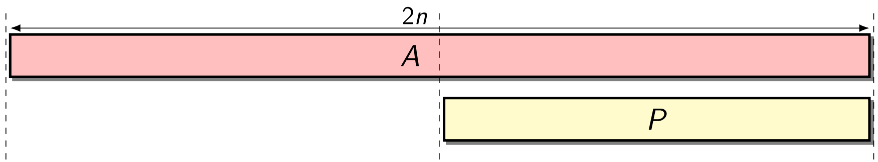

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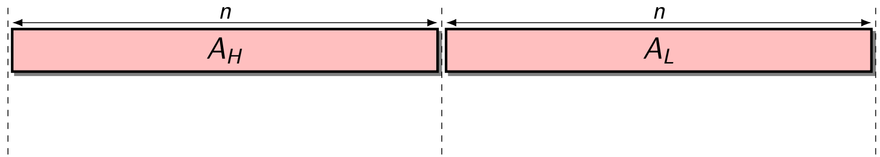

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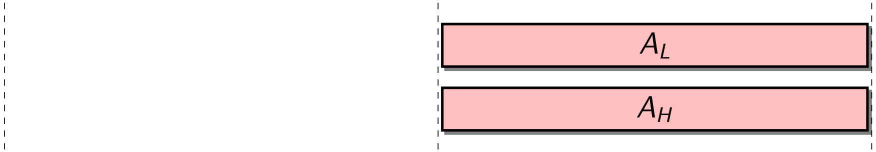

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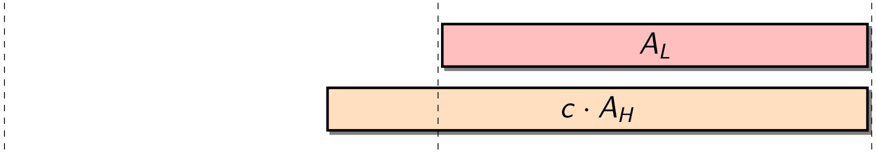

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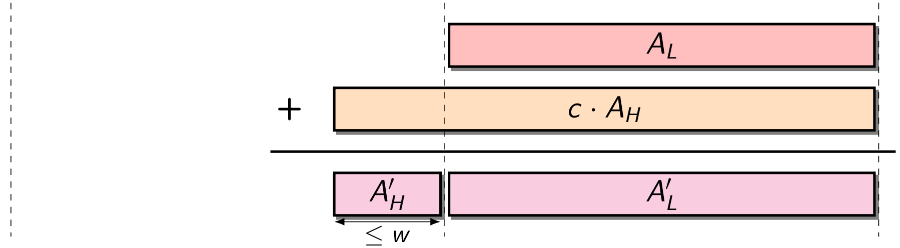

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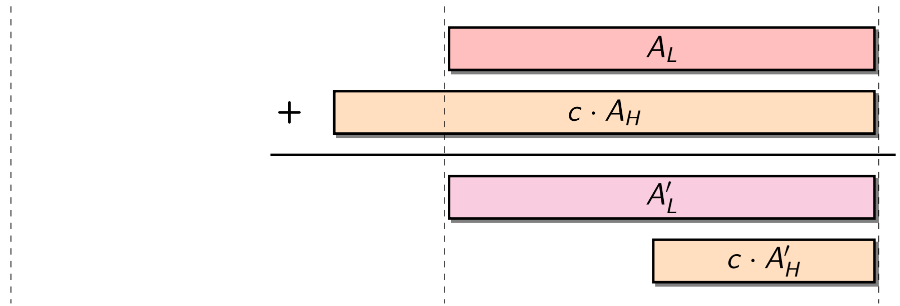

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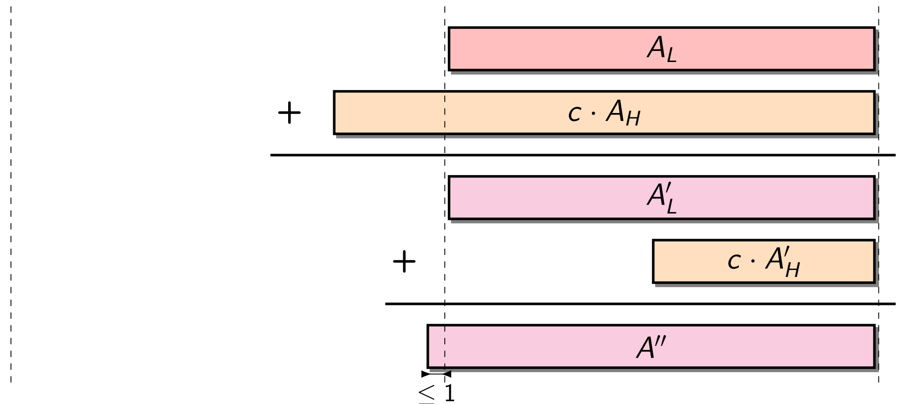

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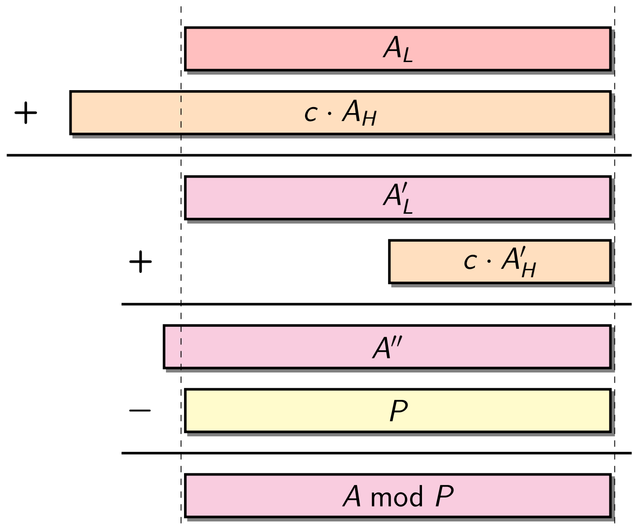

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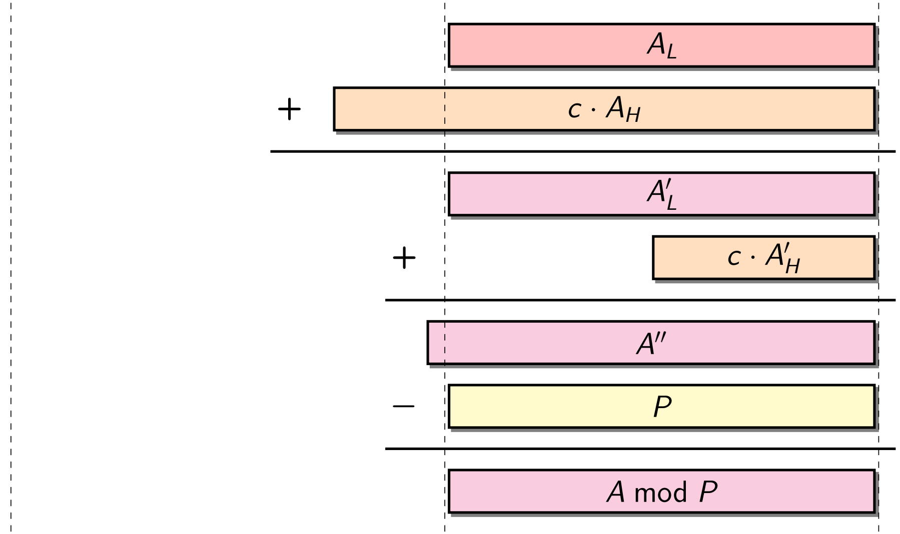
- Examples: $P = 2^{255} - 19$ (Curve25519) or $P = 2^{448} - 2^{224} - 1$ (Ed448-Goldilocks)



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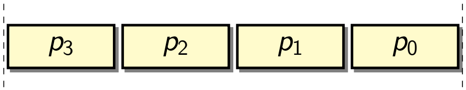

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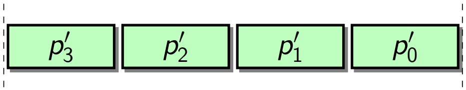

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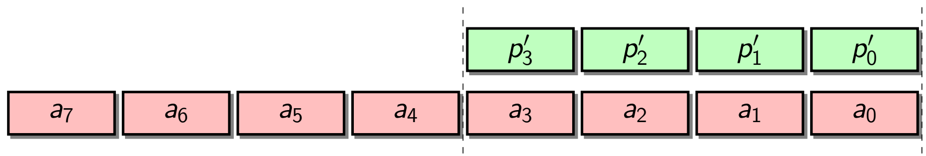

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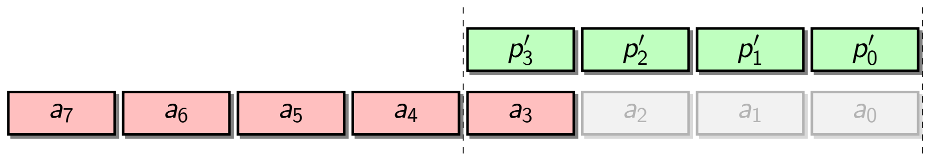

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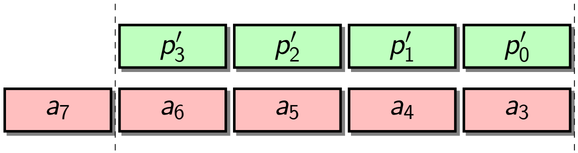

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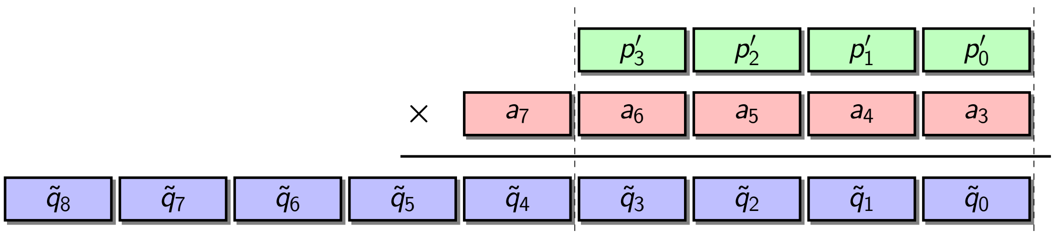

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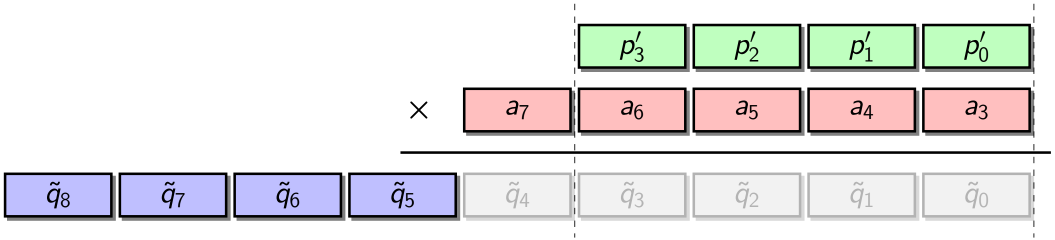

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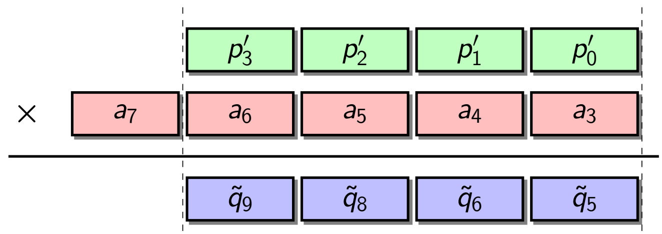

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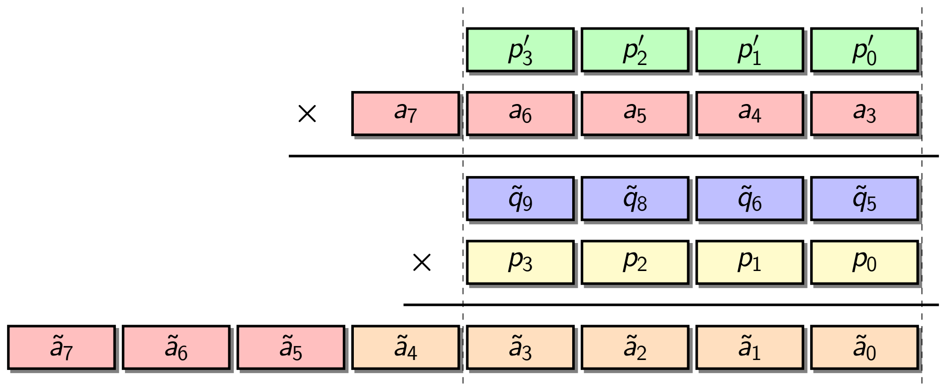

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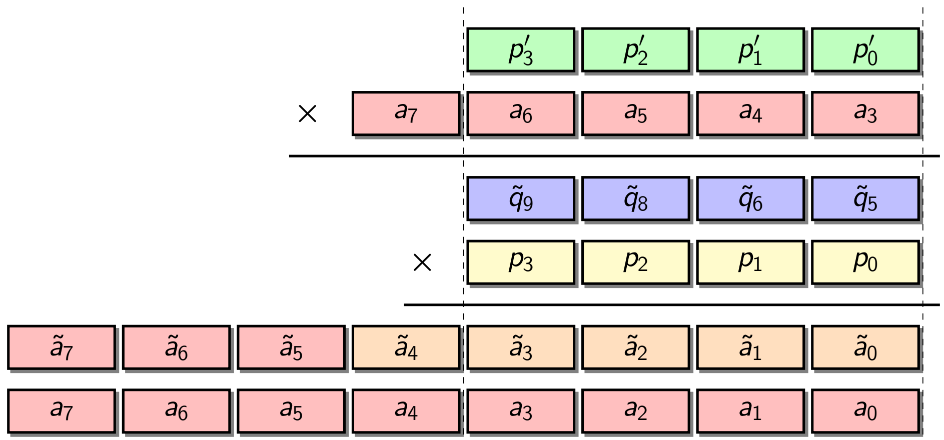

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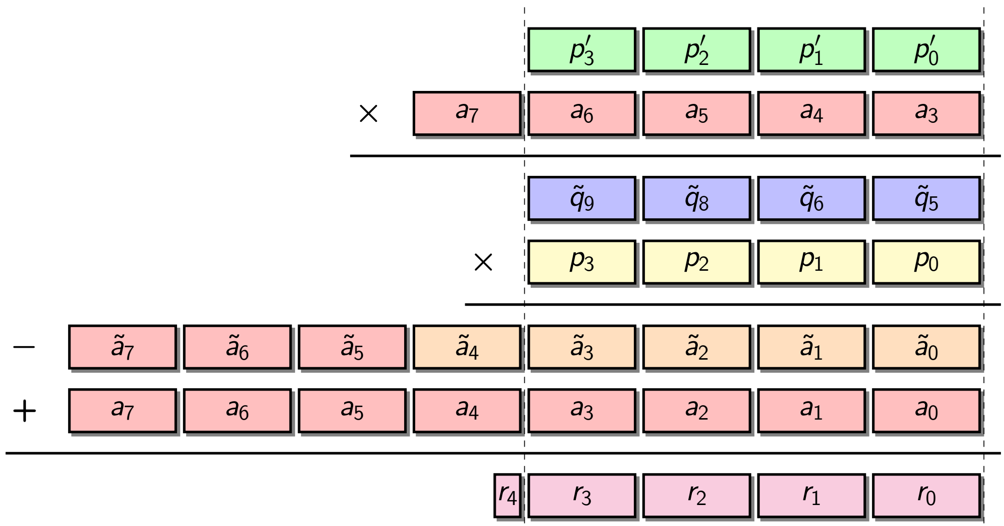

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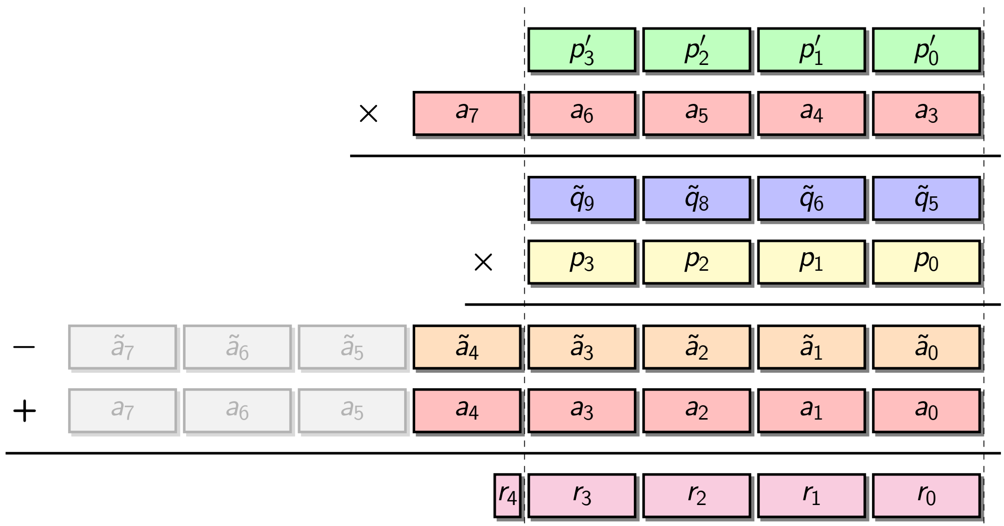

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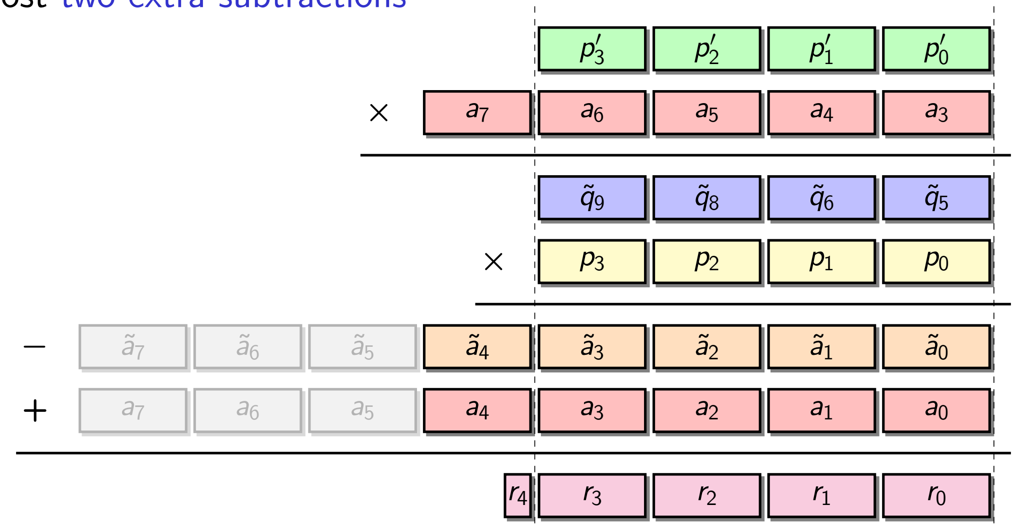

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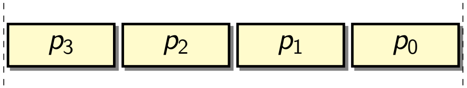

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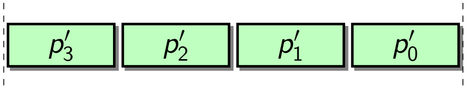

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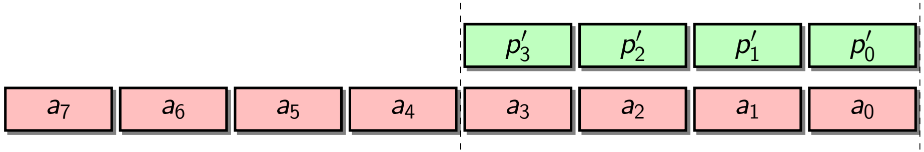

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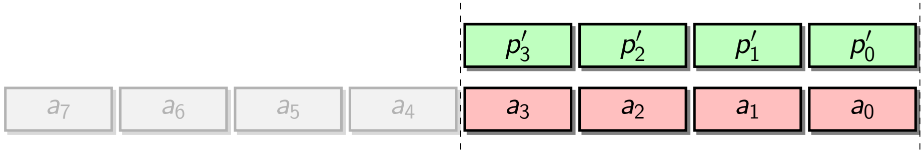

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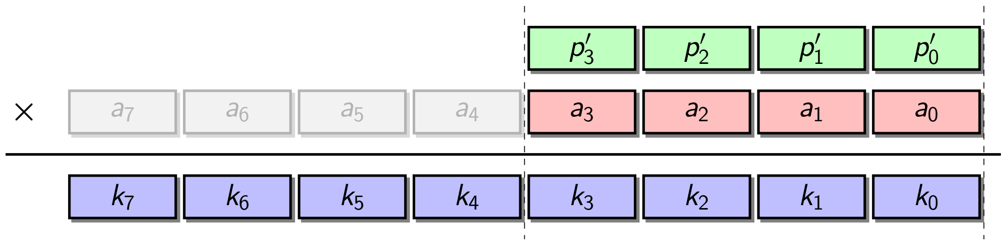

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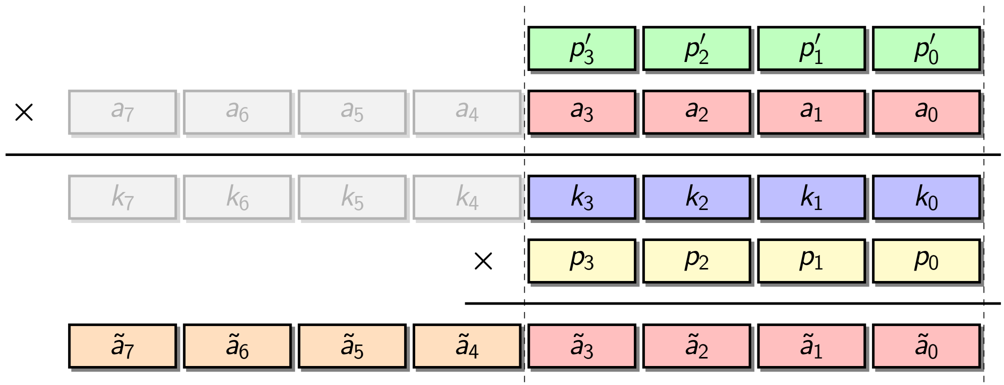

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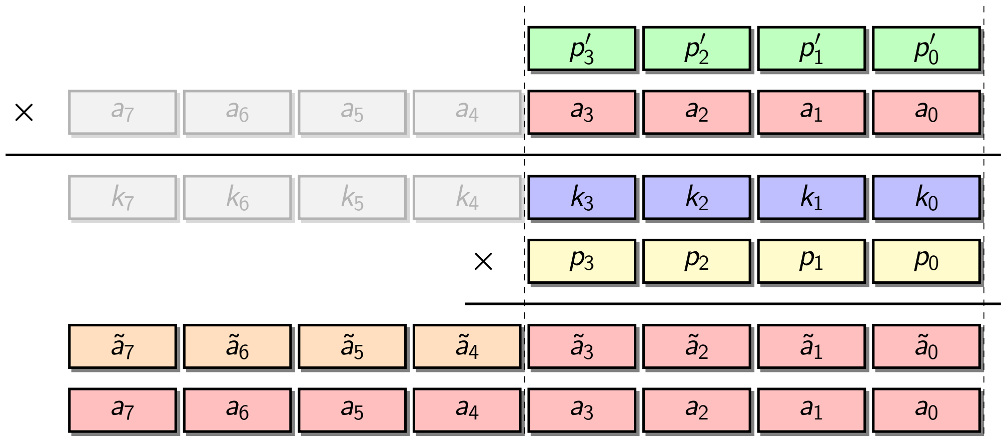

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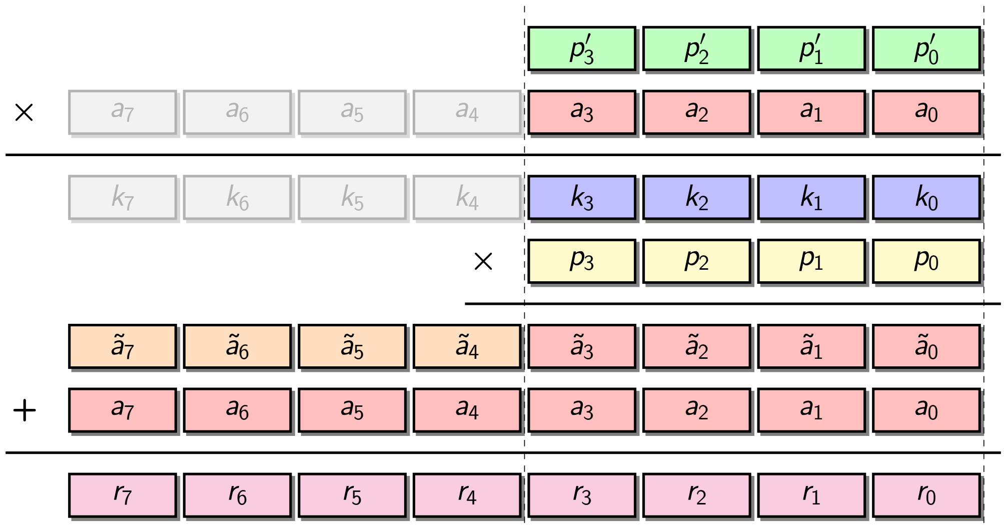

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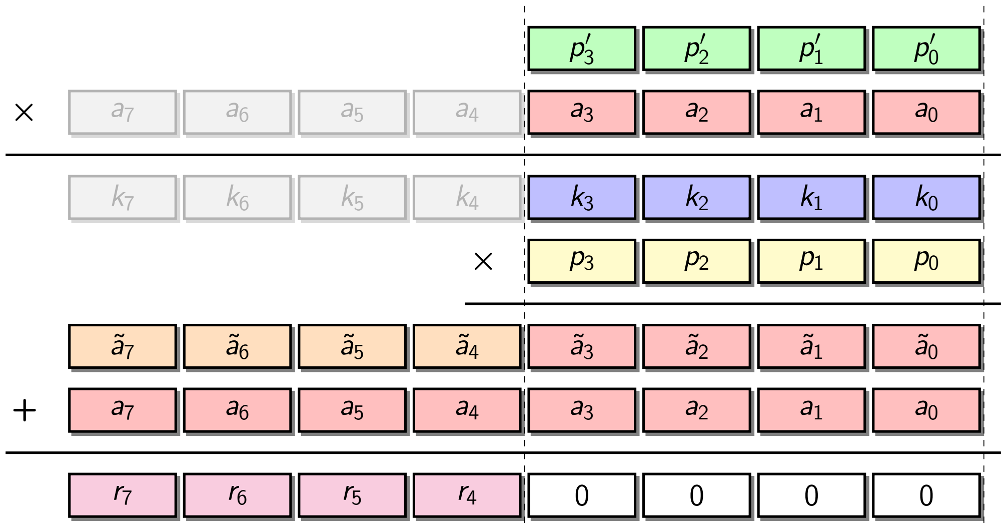

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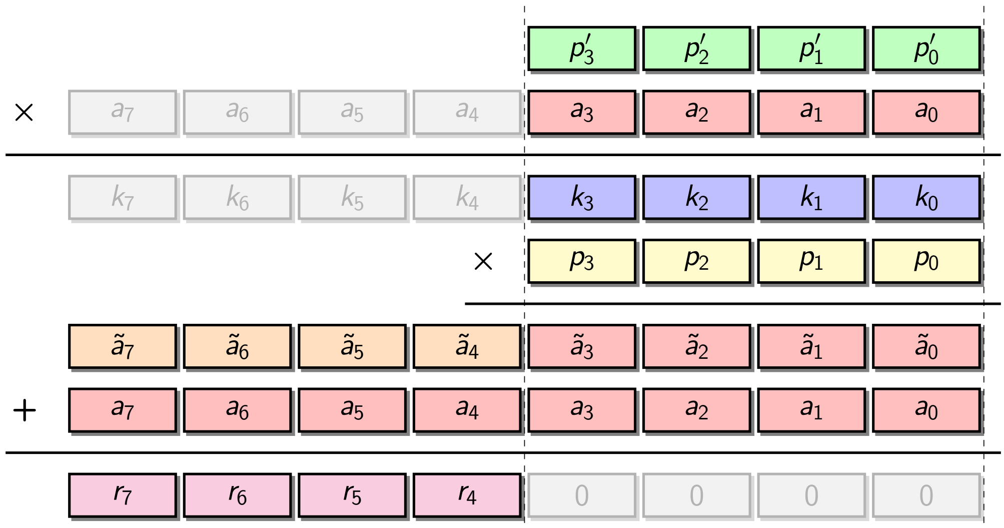

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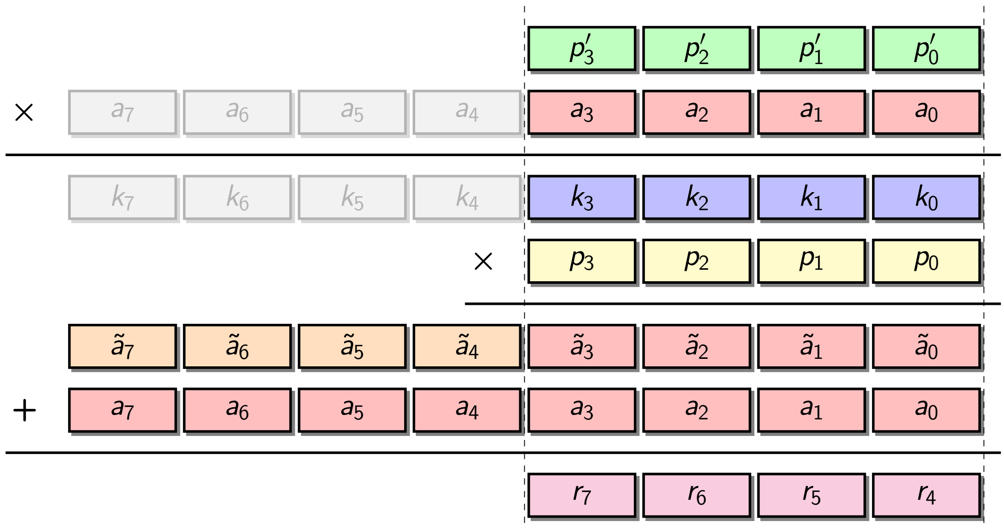

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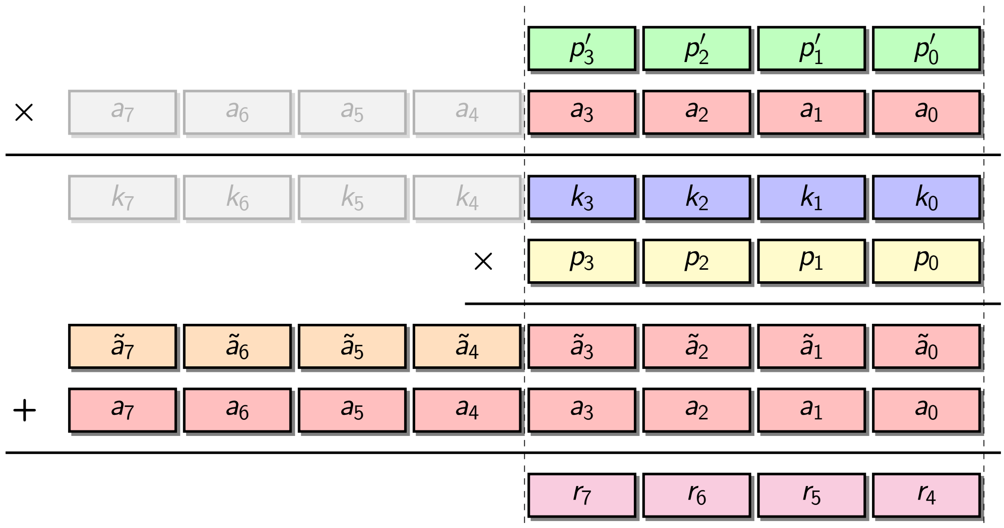

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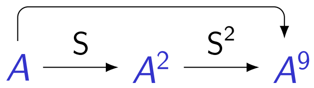

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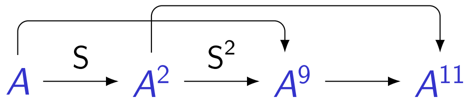

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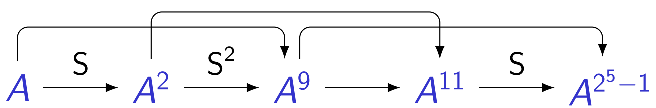

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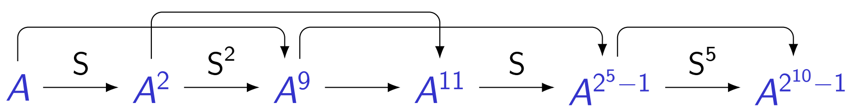

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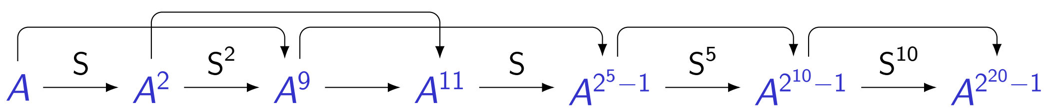

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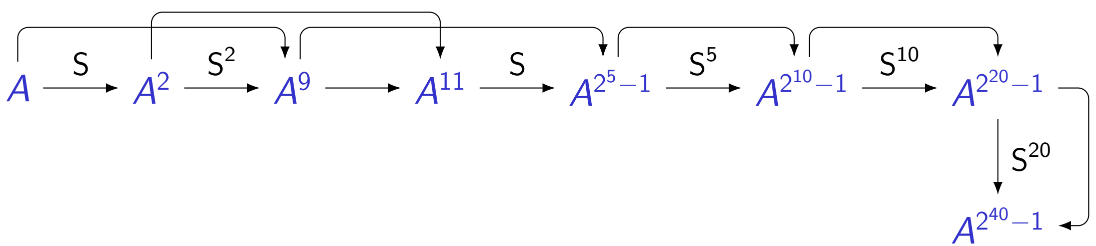

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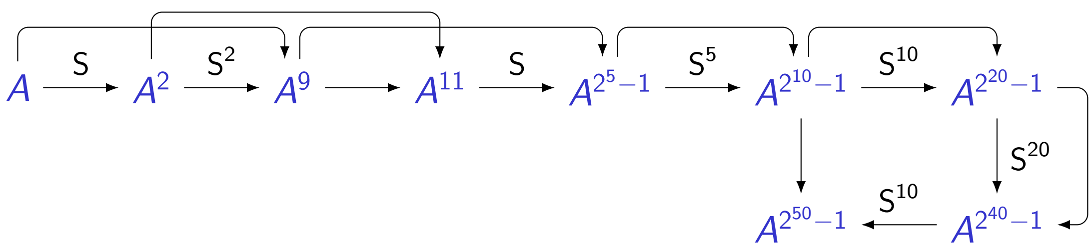

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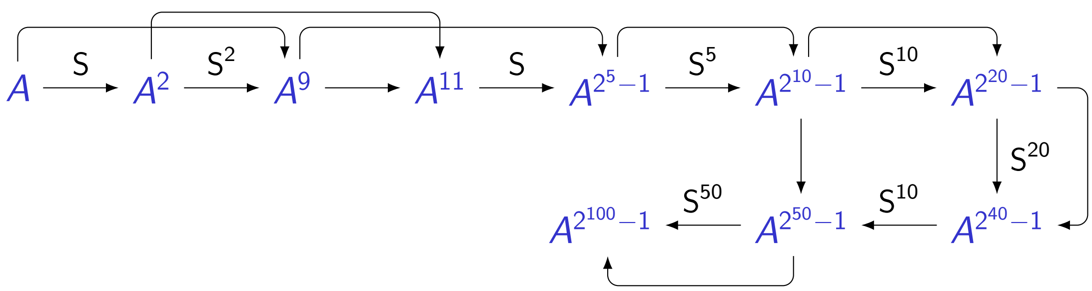

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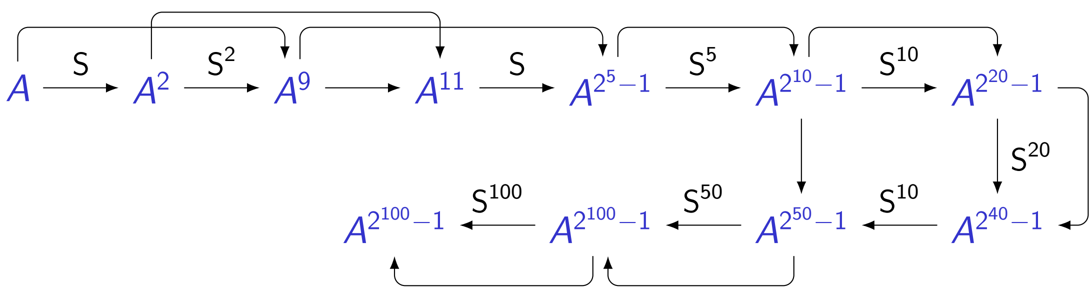

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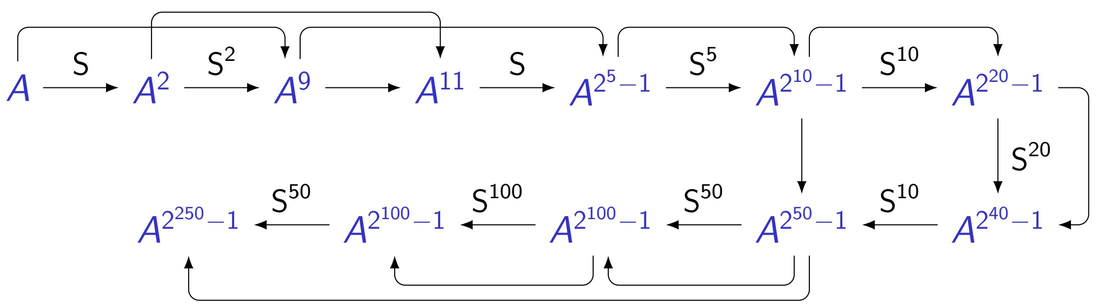

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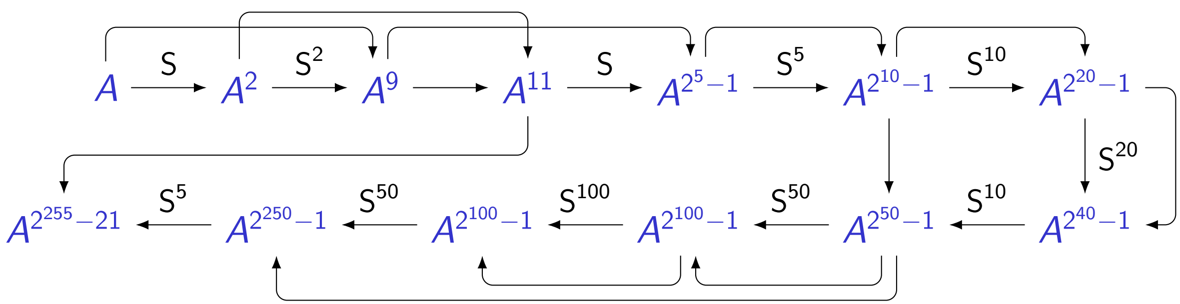

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- Let $\mathcal{B} = (m_1, \ldots, m_k)$ a tuple of k pairwise coprime integers
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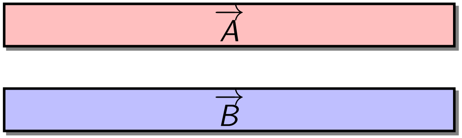

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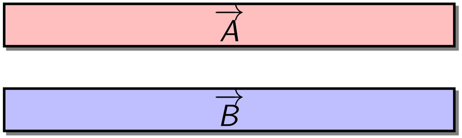

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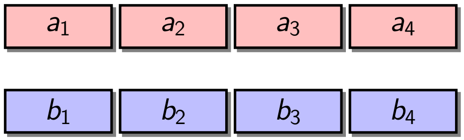

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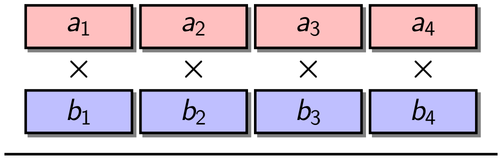

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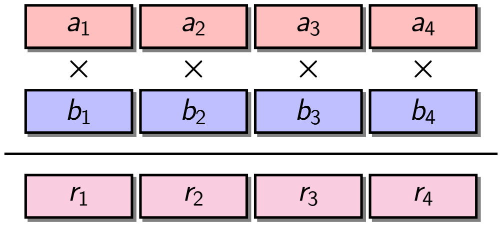

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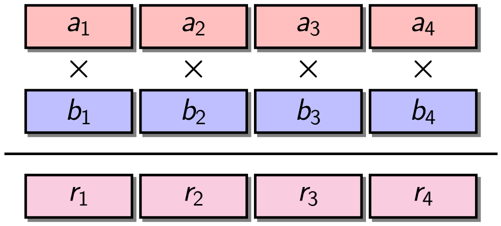

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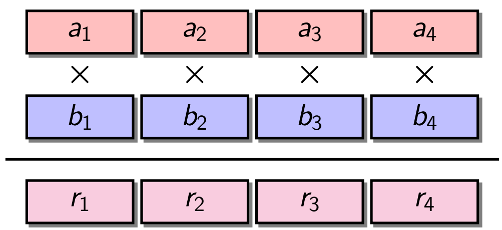
- Limitations:
  - operations are computed in $\mathbb{Z}/M\mathbb{Z}$: beware of overflows!
  - no simple way to compute divisons, modular reductions or comparisons



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  $$q = \left\lfloor \sum_{i=1}^{k} \frac{|a_i \cdot M_i^{-1}|_{m_i} \cdot M_i}{M} \right\rfloor \approx \left\lfloor \sum_{i=1}^{k} \frac{|a_i \cdot M_i^{-1}|_{m_i}}{2^w} \right\rfloor$$
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RNS modular reduction

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  $\Rightarrow$ Need to approximate CRT reconstruction and reduce it modulo P
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- If $0 \leq A < (1 - \varepsilon)M$, then $\tilde{q} = q$ and
  $$A \bmod P \equiv \left(\sum_{i=1}^{k} |a_i \cdot M_i^{-1}|_{m_i} \cdot |M_i|_P \right) - |\tilde{q}M|_P \pmod{P}$$

RNS modular reduction

$$A \bmod P \equiv \left(\sum_{i=1}^{k} |a_i \cdot M_i^{-1}|_{m_i} \cdot |M_i|_P \right) - |\tilde{q} M|_P \pmod{P}$$

function reduce-mod-$P(\overrightarrow{A})$**:**

($\forall i$**)** $z_i \leftarrow \left| a_i \cdot |M_i^{-1}|_{m_i} \right|_{m_i}$

($\forall i$**)** $\tilde{z}_i \leftarrow \lfloor z_i / 2^{w-t} \rfloor$

$\tilde{q} \leftarrow \lfloor \sum_i \tilde{z}_i / 2^t + \varepsilon \rfloor$

($\forall i$**)** $r_i \leftarrow 0$

for $j \leftarrow 1$ **to** k**:**

($\forall i$**)** $r_i \leftarrow \left| r_i + z_j \cdot \left| |M_j|_P \right|_{m_i} \right|_{m_i}$

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- Precomputations:

RNS modular reduction

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- Precomputations:
  - for all $i \in \{1, \ldots, k\}$, $|M_i^{-1}|_{m_i}$ (k words)

RNS modular reduction

$$A \bmod P \equiv \left(\sum_{i=1}^{k} |a_i \cdot M_i^{-1}|_{m_i} \cdot |M_i|_P \right) - |\tilde{q}M|_P \pmod{P}$$

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  - for all $i \in \{1, \ldots, k\}$, $\overrightarrow{|M_i^{-1}|_{m_i}}$ (k words)
  - for all $j \in \{1, \ldots, k\}$, $\overrightarrow{|M_j|_P}$ (k^2 words)

RNS modular reduction

$$A \bmod P \equiv \left(\sum_{i=1}^{k} |a_i \cdot M_i^{-1}|_{m_i} \cdot |M_i|_P \right) - |\tilde{q}M|_P \pmod{P}$$

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  - for all $\tilde{q} \in \{1, \ldots, k-1\}$, $\overrightarrow{|\tilde{q}M|_P}$ (k^2 words)

RNS modular reduction

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- Cost:

RNS modular reduction

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- Cost: k mults

RNS modular reduction

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- Precomputations:
  - for all $i \in \{1, \ldots, k\}$, $|M_i^{-1}|_{m_i}$ (k words)
  - for all $j \in \{1, \ldots, k\}$, $\overrightarrow{|M_j|_P}$ (k^2 words)
  - for all $\tilde{q} \in \{1, \ldots, k-1\}$, $\overrightarrow{|\tilde{q}M|_P}$ (k^2 words)
- Cost: k mults + k^2 mults

RNS modular reduction

$$A \bmod P \equiv \left(\sum_{i=1}^{k} |a_i \cdot M_i^{-1}|_{m_i} \cdot |M_i|_P \right) - |\tilde{q} M|_P \pmod{P}$$

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  - for all $\tilde{q} \in \{1, \ldots, k-1\}$, $\overrightarrow{|\tilde{q} M|_P}$ (k^2 words)
- Cost: k mults + k^2 mults → quadratic complexity

RNS Montgomery reduction

- Requires two RNS bases $\mathcal{B}_\alpha = (m_{\alpha,1}, \ldots, m_{\alpha,k})$ and $\mathcal{B}_\beta = (m_{\beta,1}, \ldots, m_{\beta,k})$ such that $P < M_\alpha$, $P < M_\beta$, and $\gcd(M_\alpha, M_\beta) = 1$

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- RNS base extension algorithm (BE) [Kawamura *et al.*, 2000]
  - given $\overrightarrow{X_\alpha}$ in base $\mathcal{B}_\alpha$, $\mathrm{BE}(\overrightarrow{X_\alpha}, \mathcal{B}_\alpha, \mathcal{B}_\beta)$ computes $\overrightarrow{X_\beta}$, the repr. of X in base $\mathcal{B}_\beta$
  - similarly, $\mathrm{BE}(\overrightarrow{X_\beta}, \mathcal{B}_\beta, \mathcal{B}_\alpha)$ computes $\overrightarrow{X_\alpha}$ in base $\mathcal{B}_\alpha$

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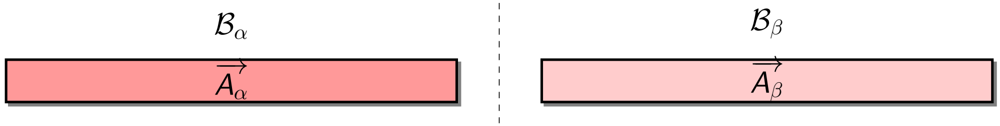
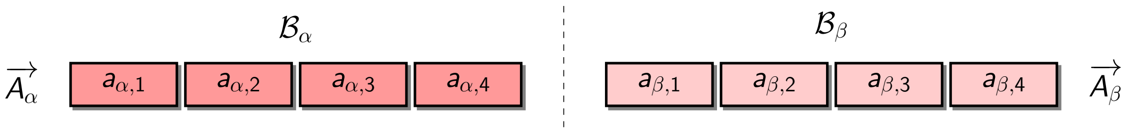
  - similarly, $\mathrm{BE}(\overrightarrow{X_\beta}, \mathcal{B}_\beta, \mathcal{B}_\alpha)$ computes $\overrightarrow{X_\alpha}$ in base $\mathcal{B}_\alpha$
  - similar to RNS modular reduction $\rightarrow$ $O(k^2)$ complexity

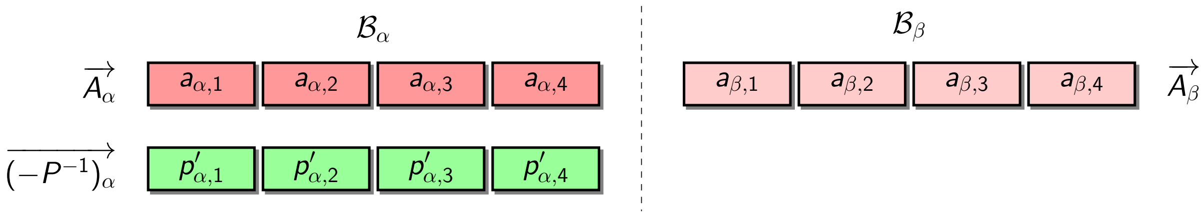
RNS Montgomery reduction



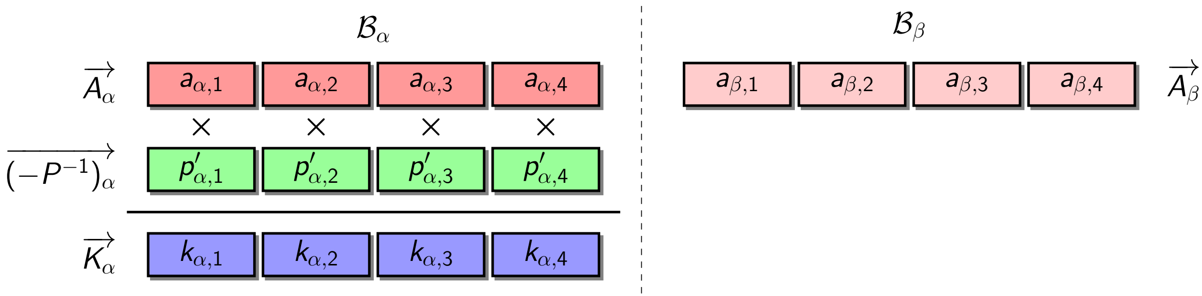
RNS Montgomery reduction



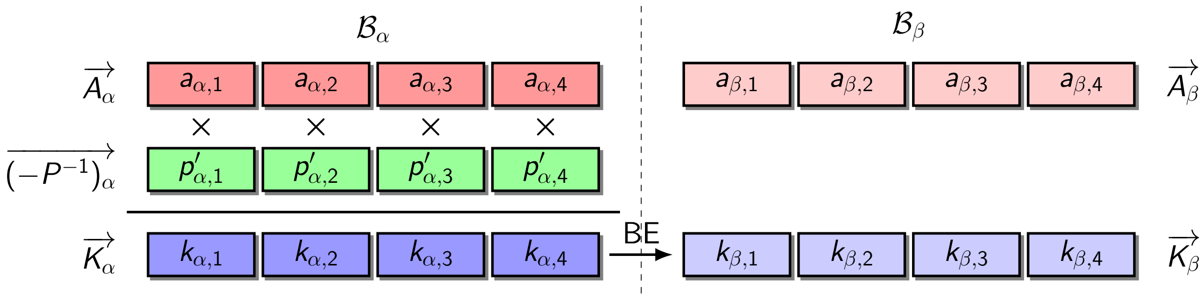
RNS Montgomery reduction



RNS Montgomery reduction



RNS Montgomery reduction



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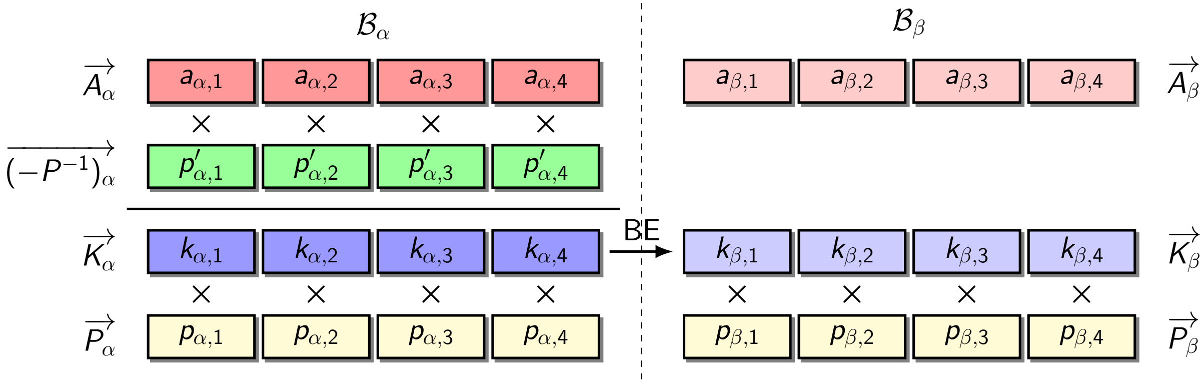
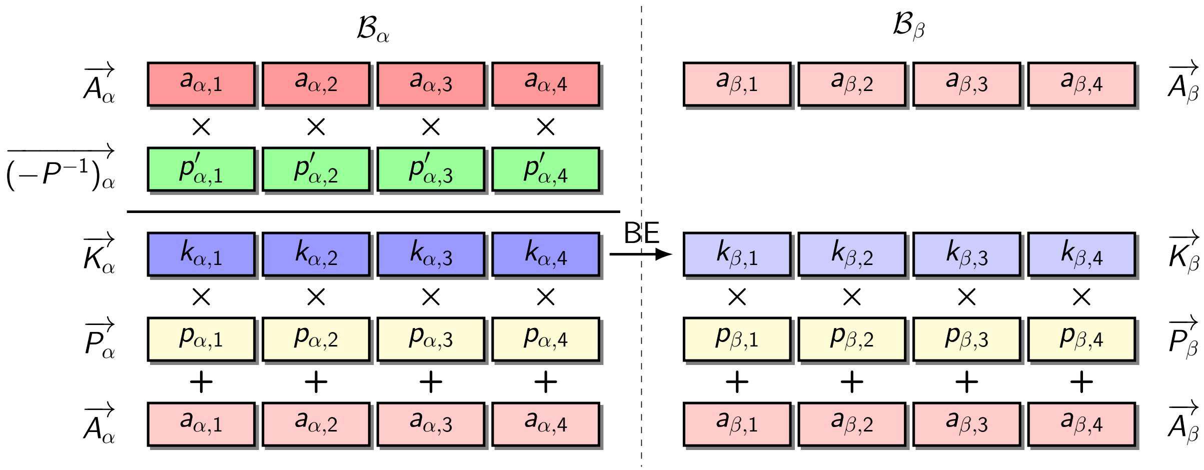
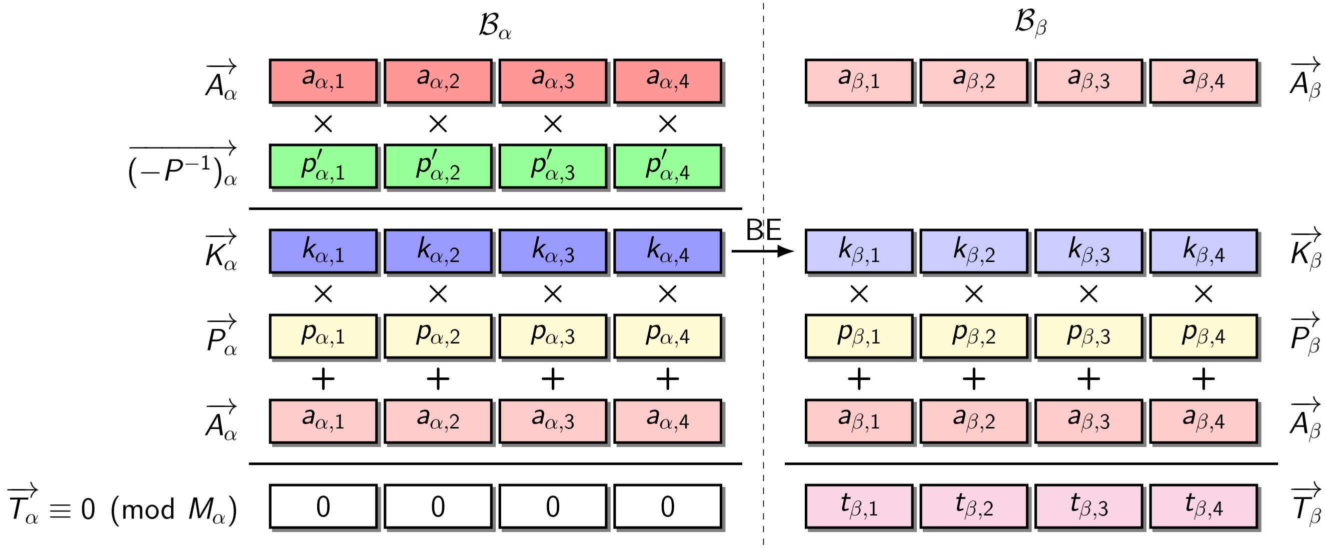


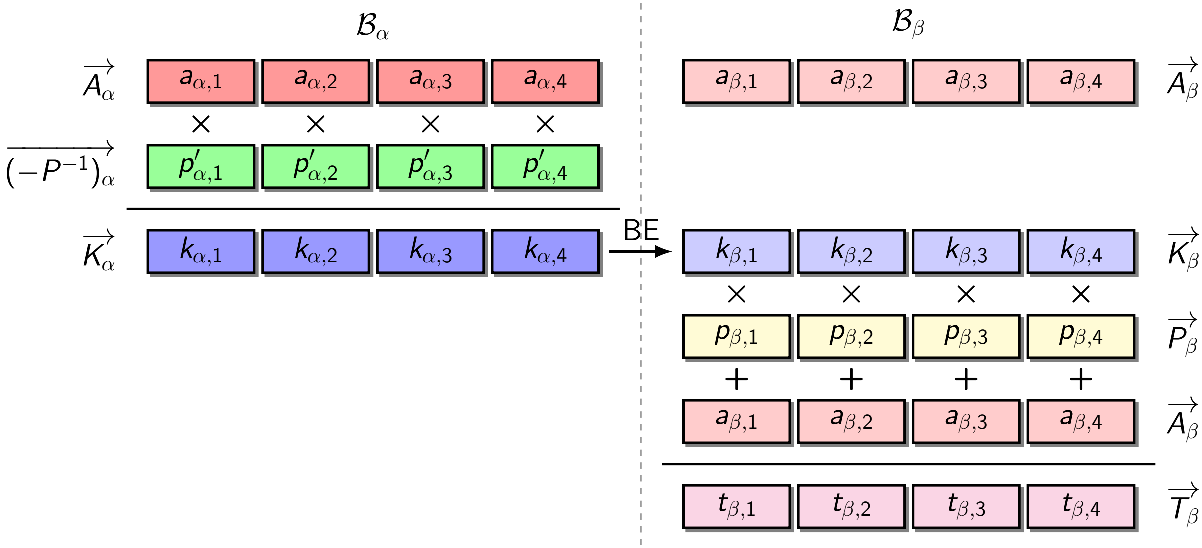
RNS Montgomery reduction



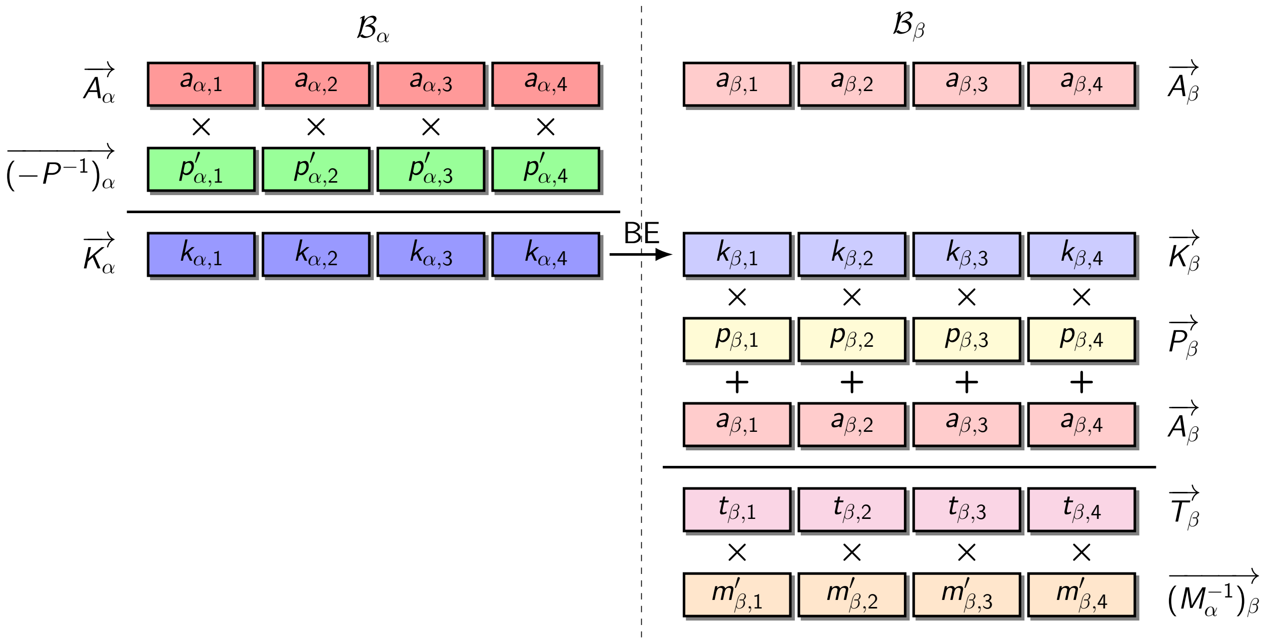
RNS Montgomery reduction



RNS Montgomery reduction



RNS Montgomery reduction



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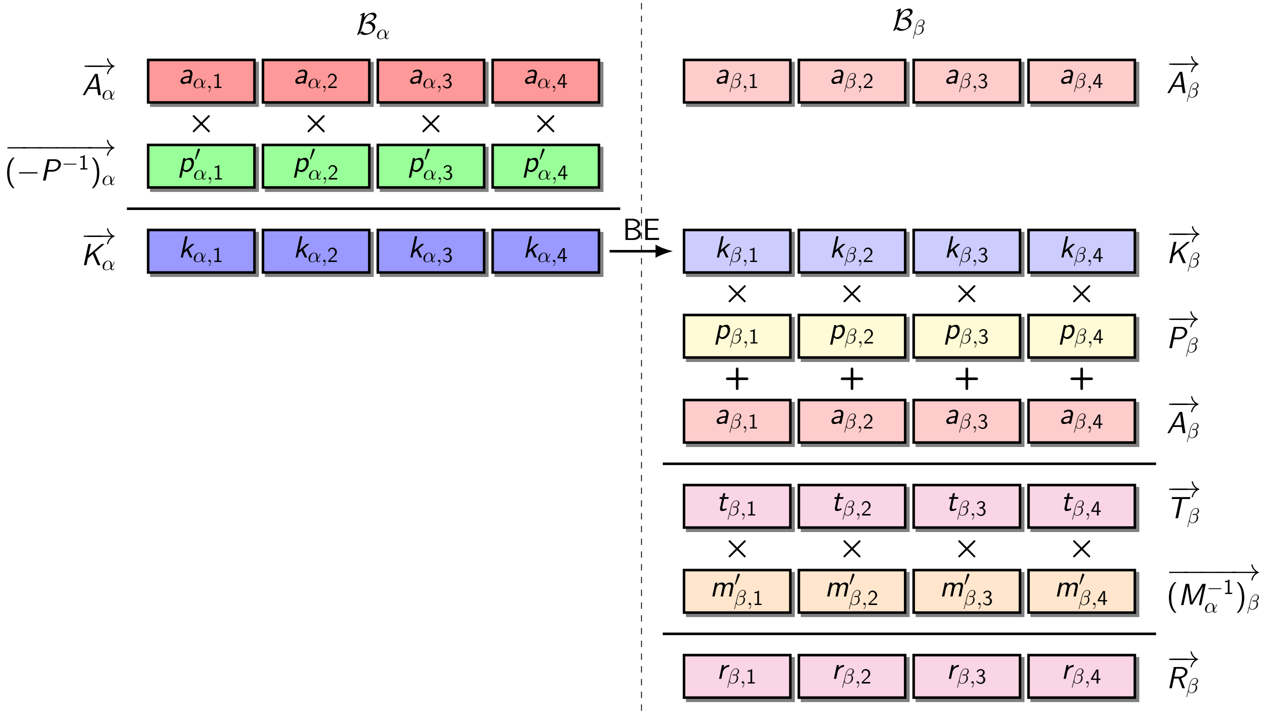

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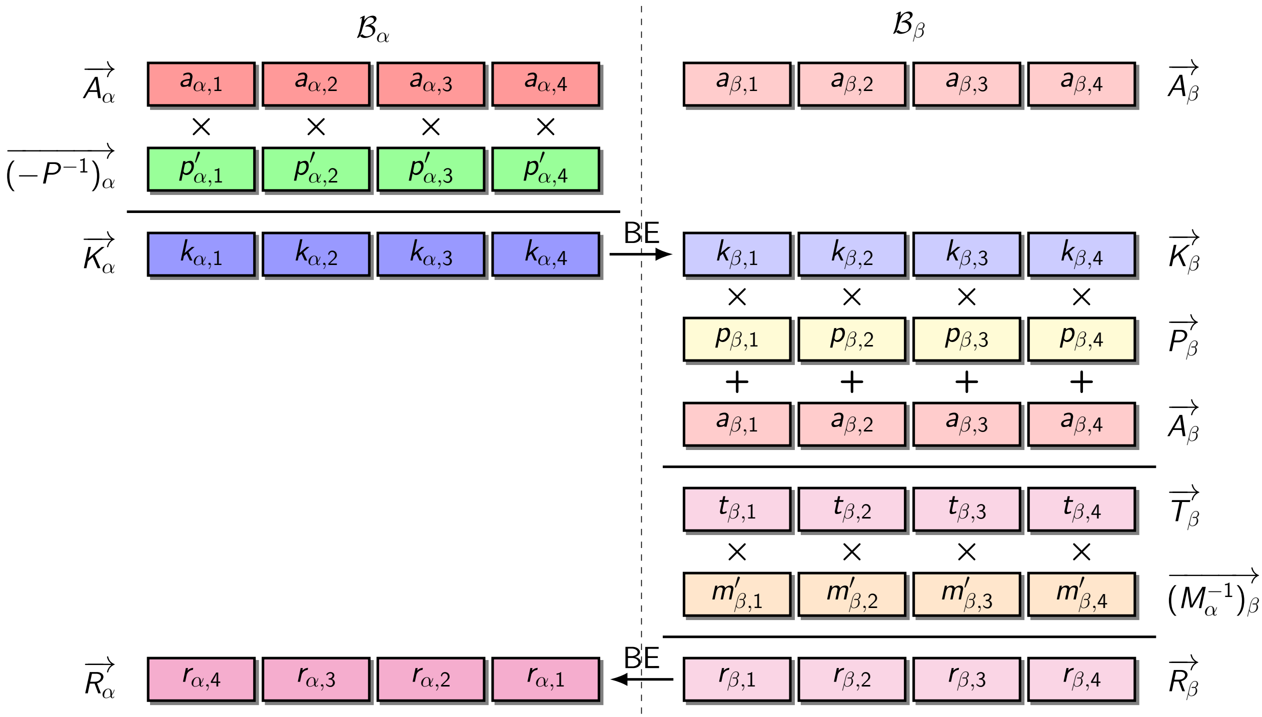

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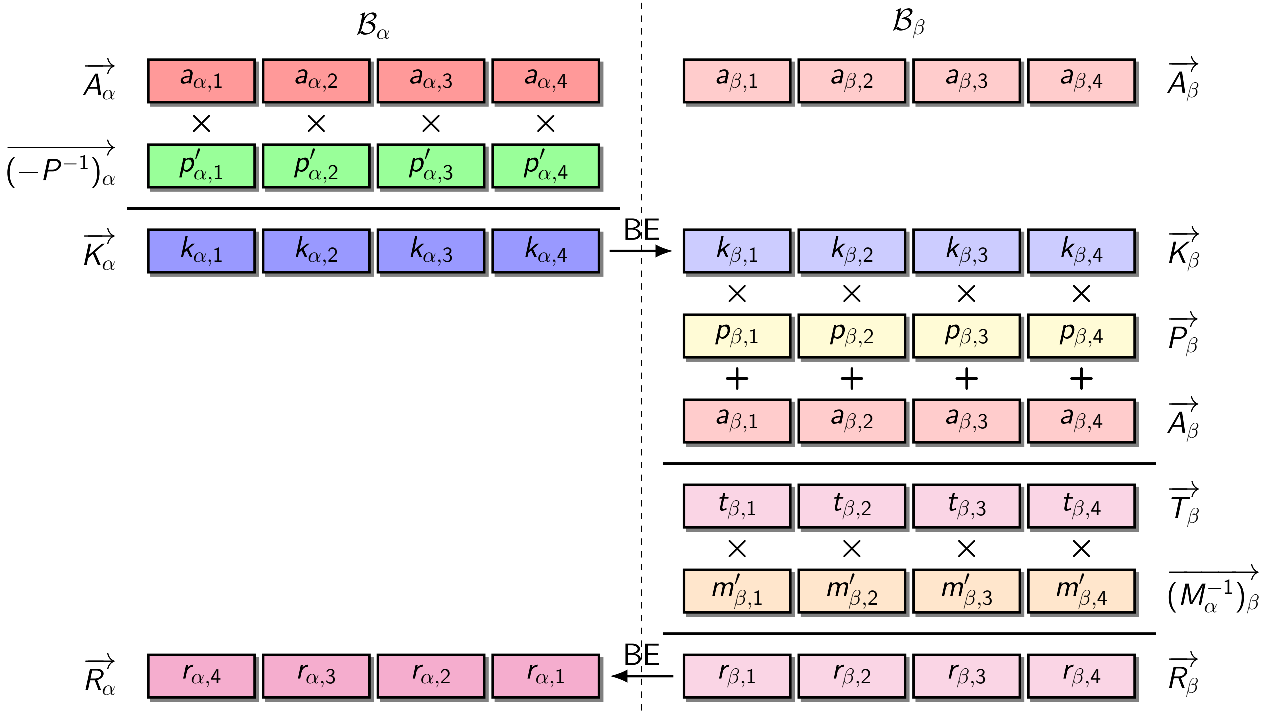

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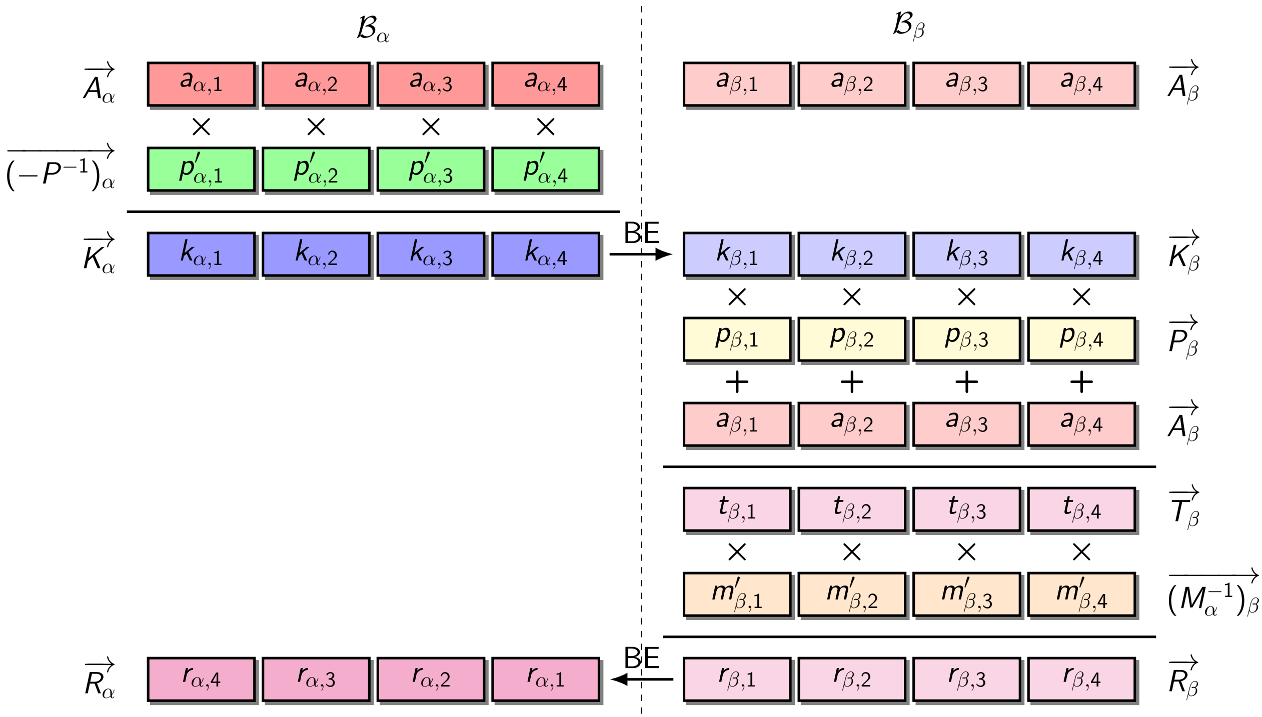

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- See recent results on this topic by Bigou and Tisserand

Outline

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- Read, code, hack, experiment!

Outline

I. Scalar multiplication

II. Elliptic curve arithmetic

III. Finite field arithmetic

IV. Software considerations

V. Notions of hardware design

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- HDL paradigm completely different from software programming languages
  - used to describe concurrent systems: unable to express sequentiality
  - structural and hierarchical description of the circuit

A half-adder in VHDL

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library ieee;
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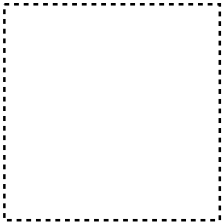

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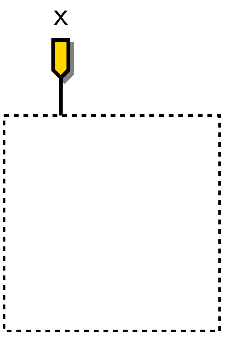

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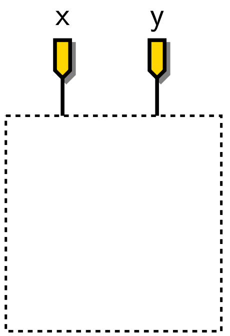

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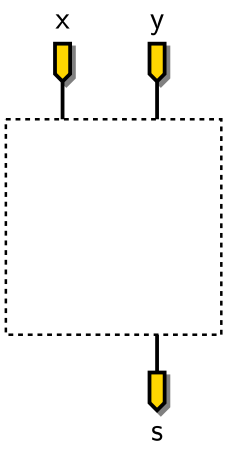

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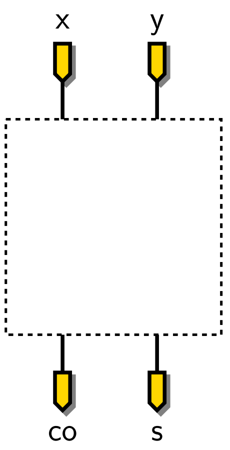

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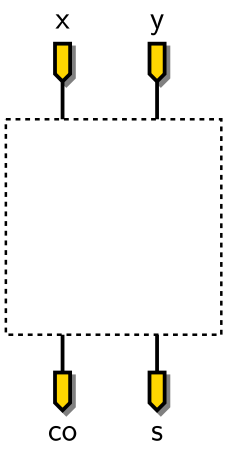

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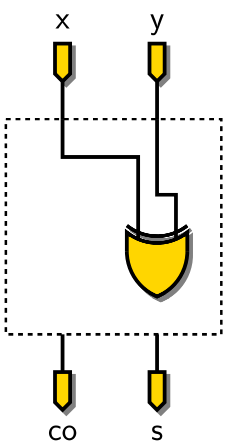

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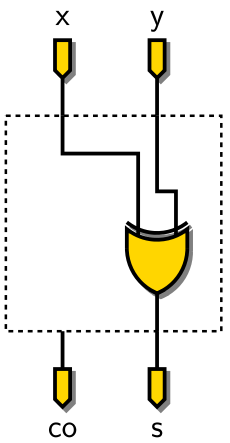

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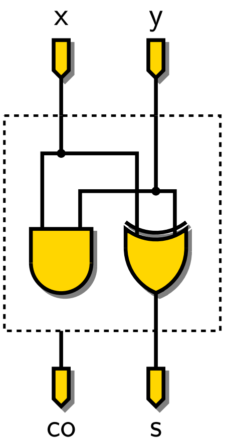

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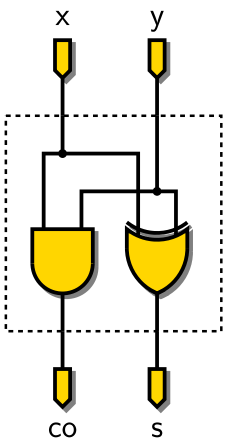

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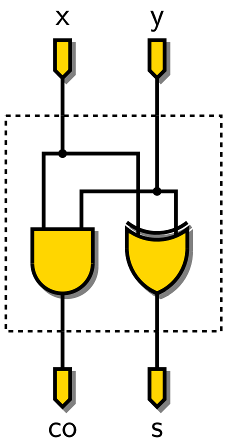

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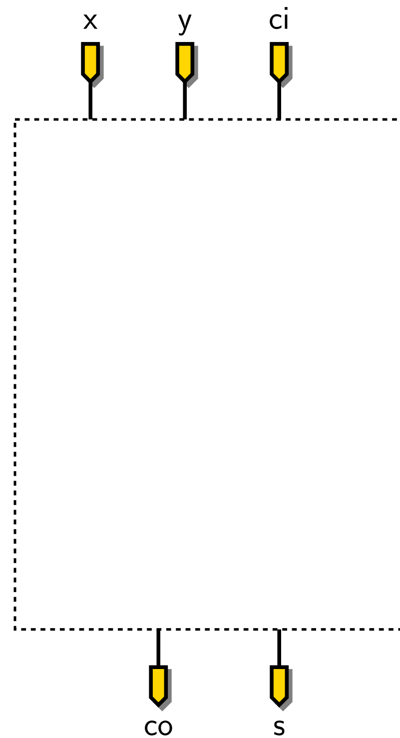

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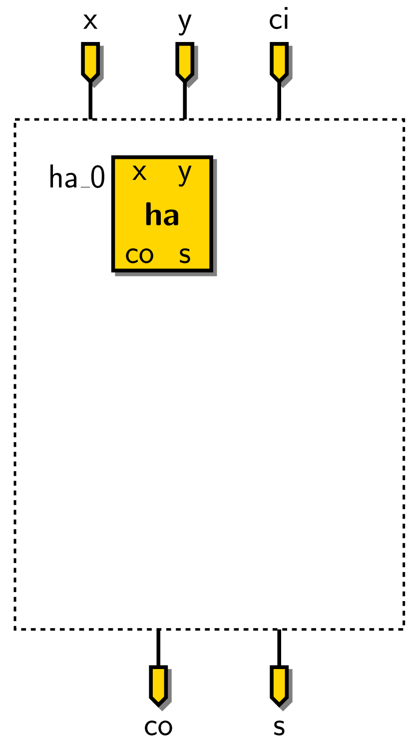

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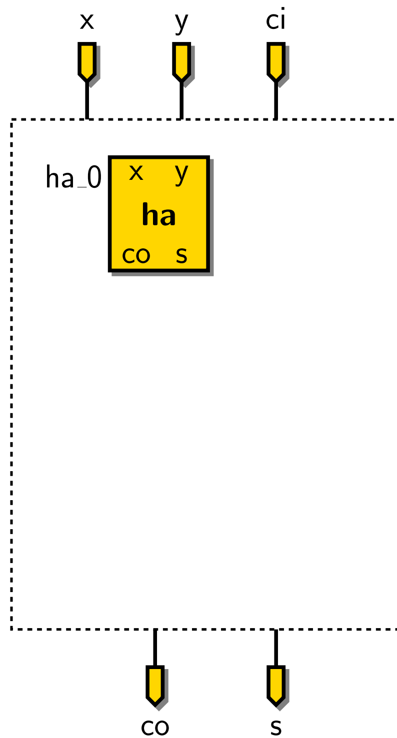

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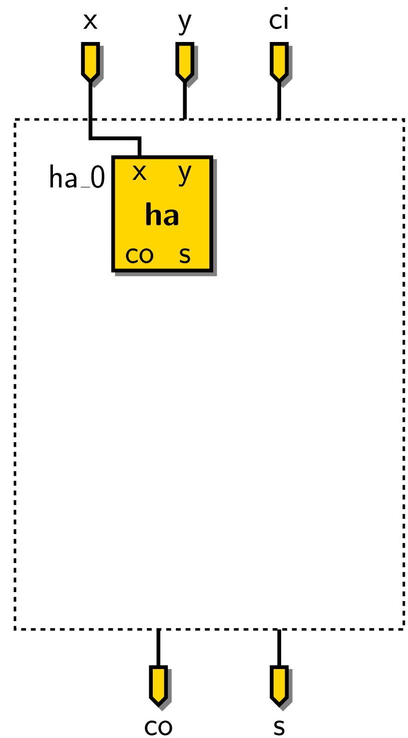

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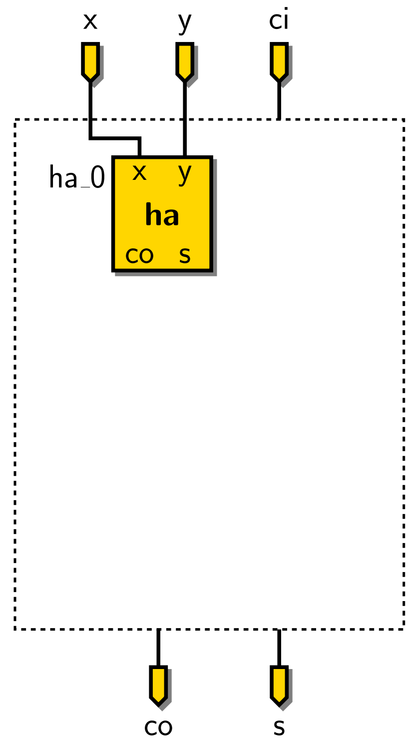

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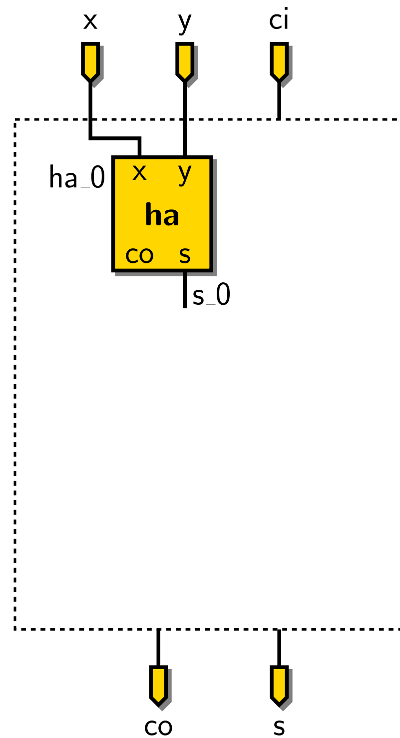

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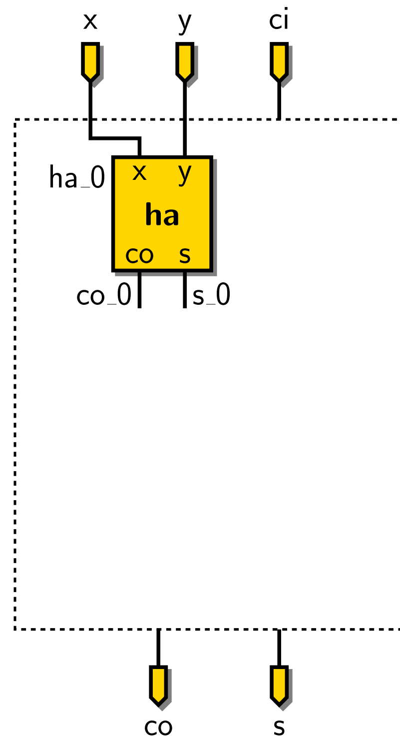

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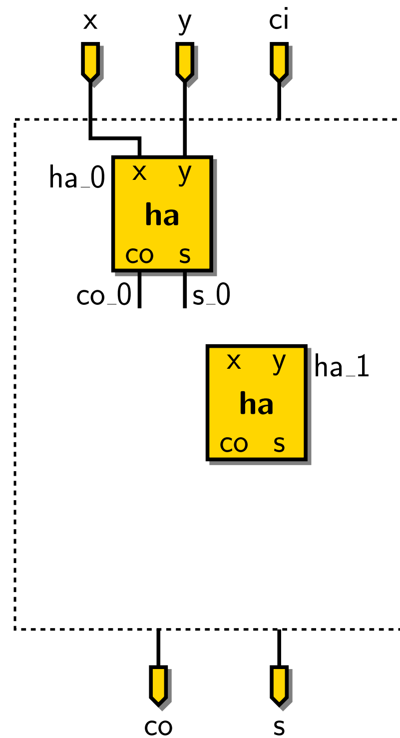

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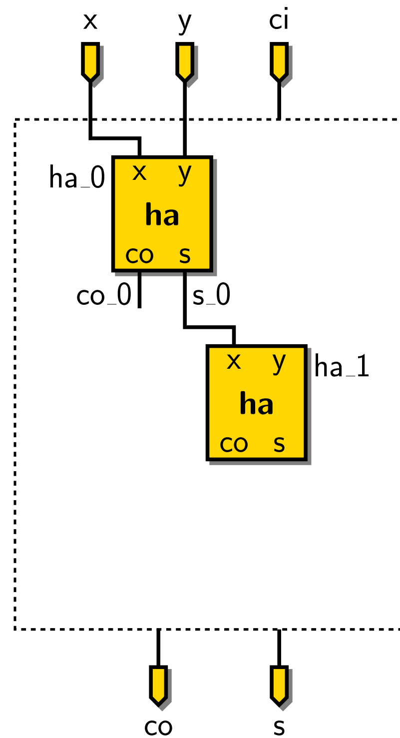

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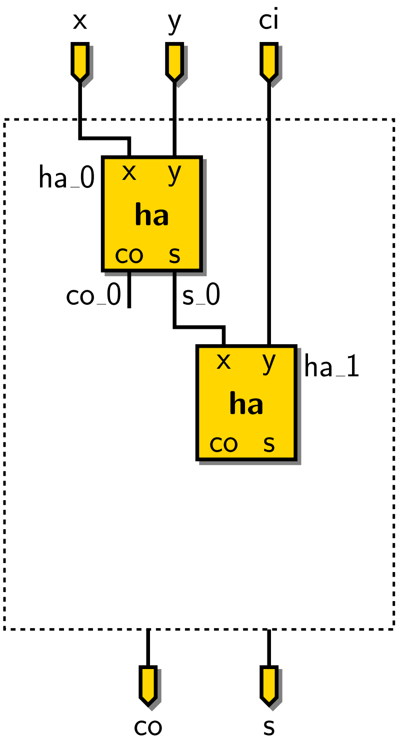

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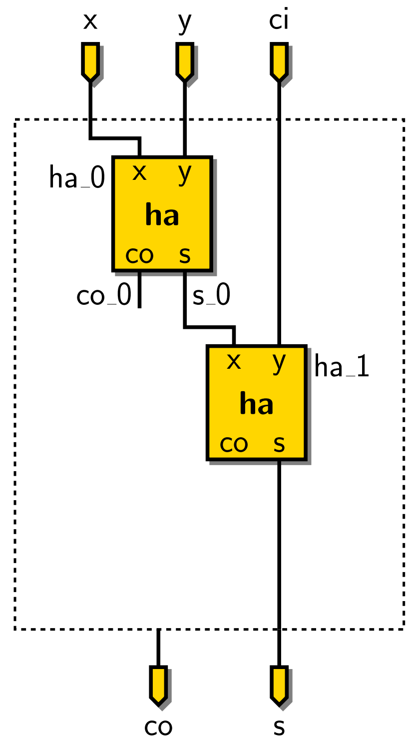

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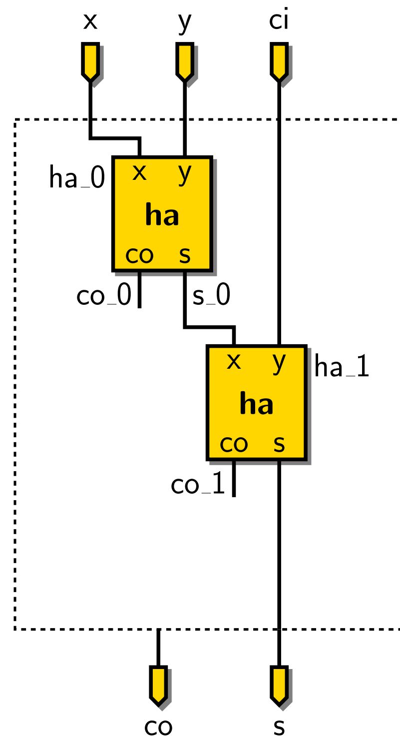

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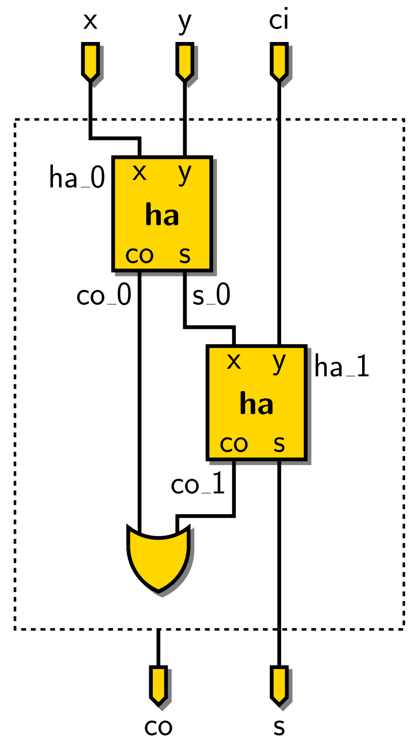

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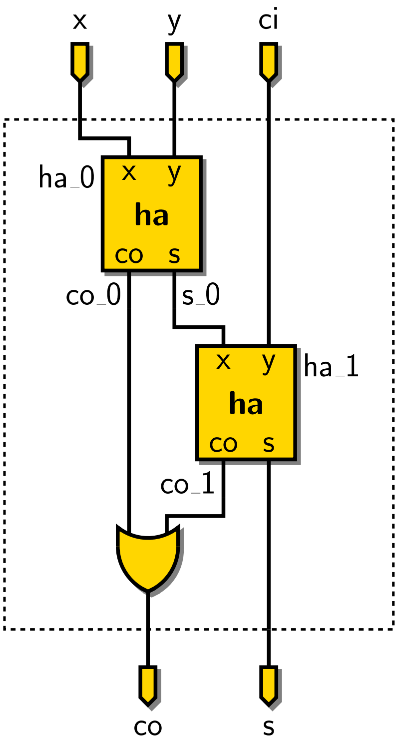

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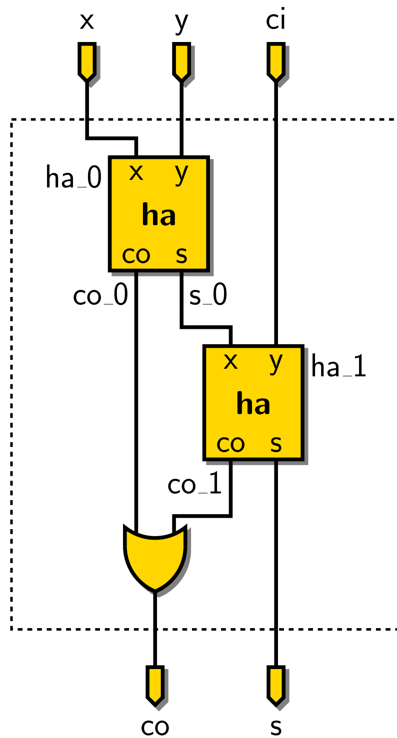

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- Implementation
  - mapping: builds a netlist of technology-dependent logic cells / transistors
  - place and route: place each logic cell on the chip and route wires between them

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  - computing $A^{2^{m-1}-1}$ only requires multiplications and Frobeniuses [Itoh and Tsujii, 1988]
  - no extra hardware for inversion

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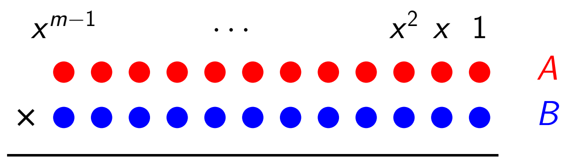

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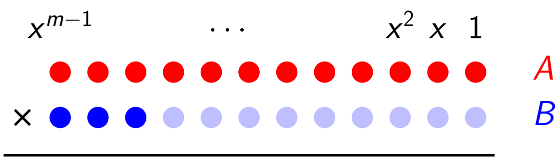

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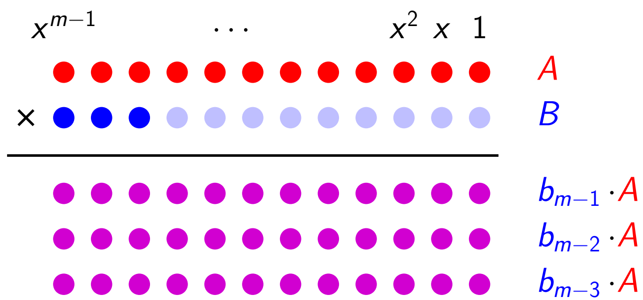

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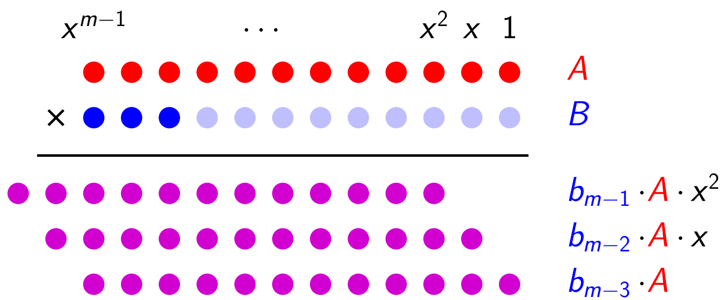

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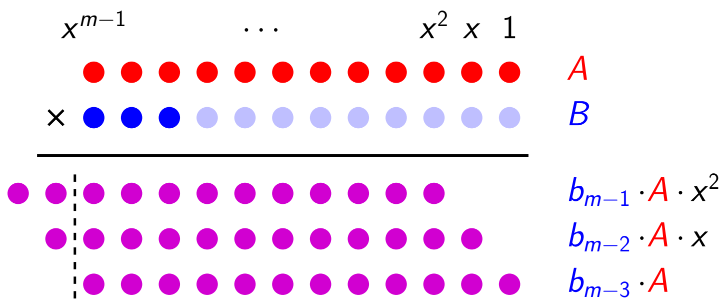

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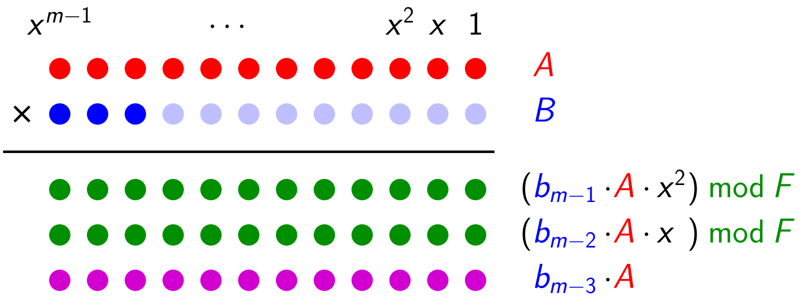

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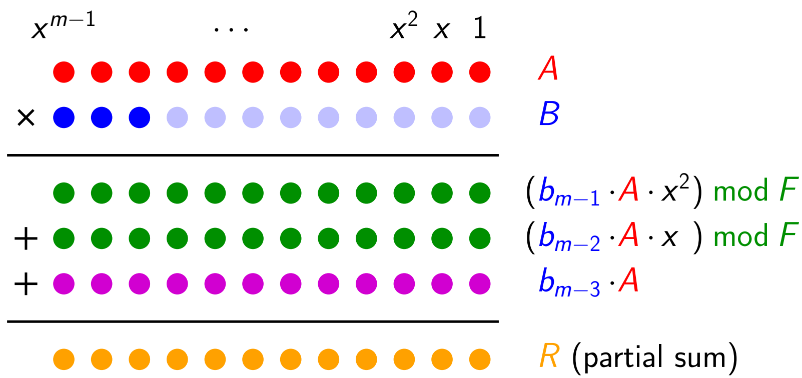

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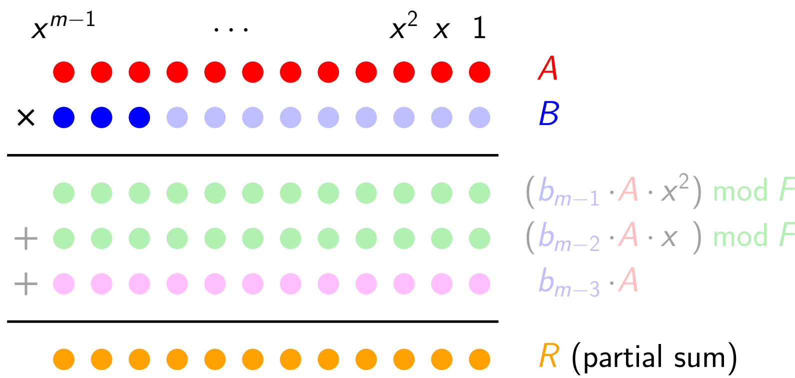

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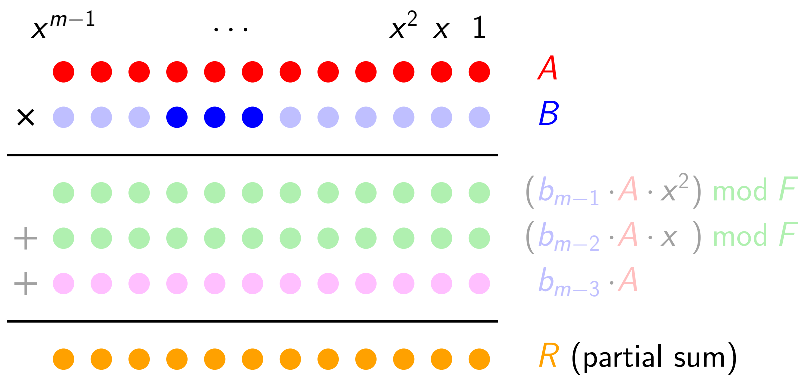

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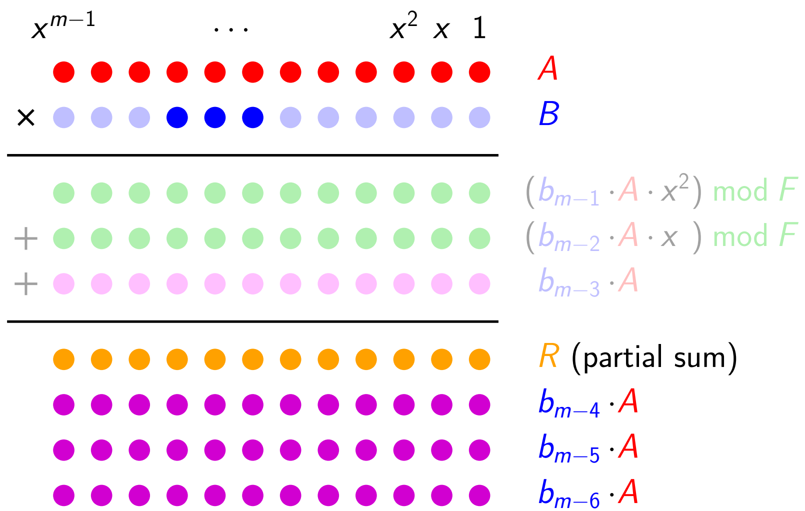

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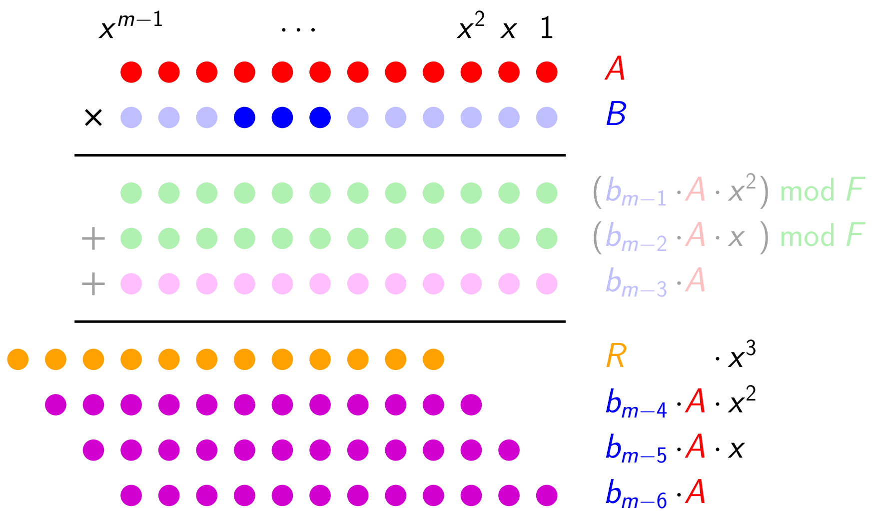

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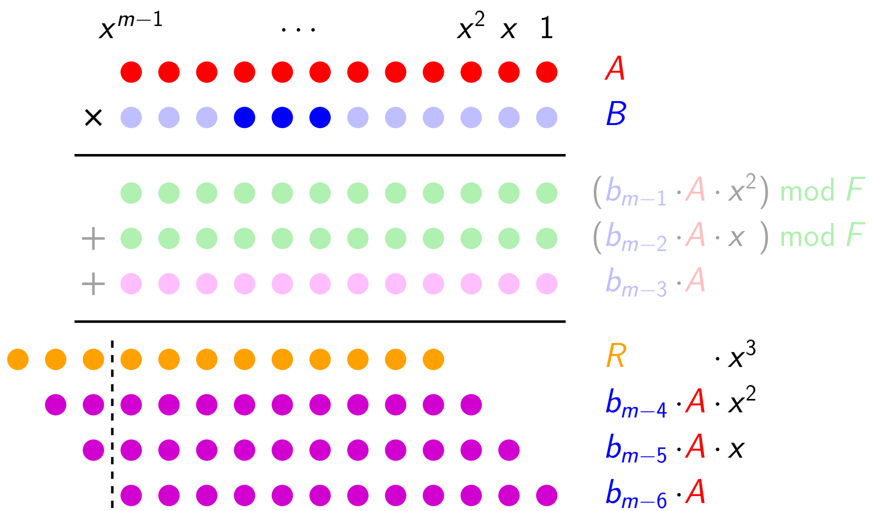

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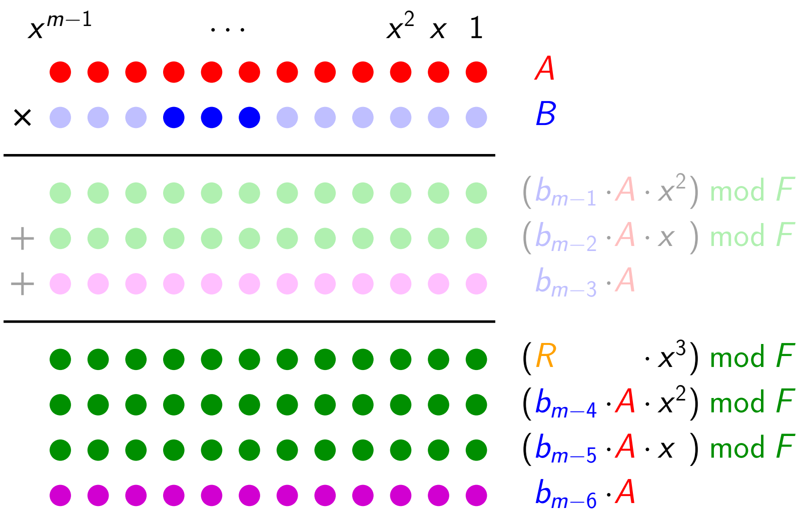

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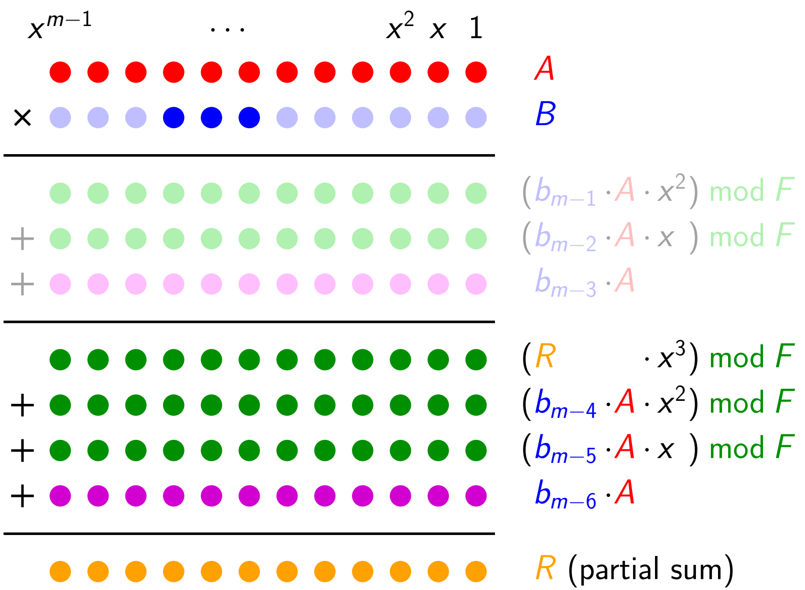

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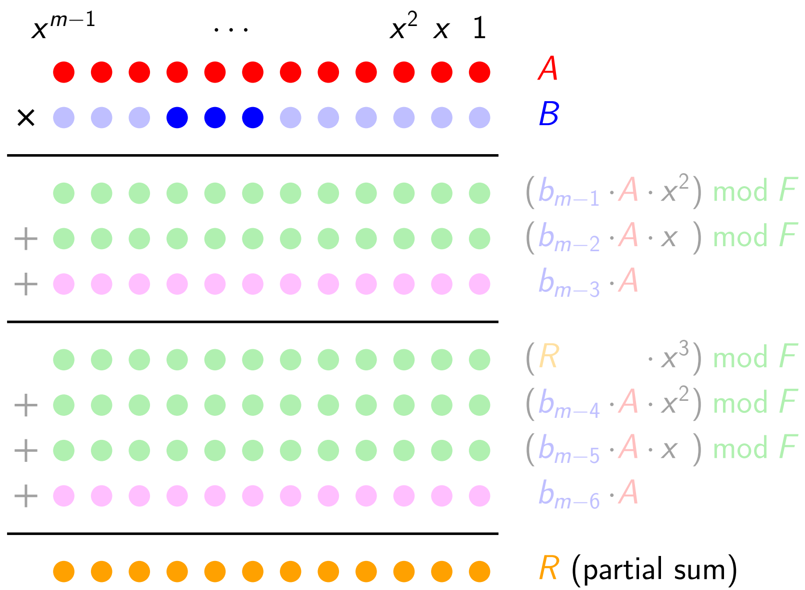

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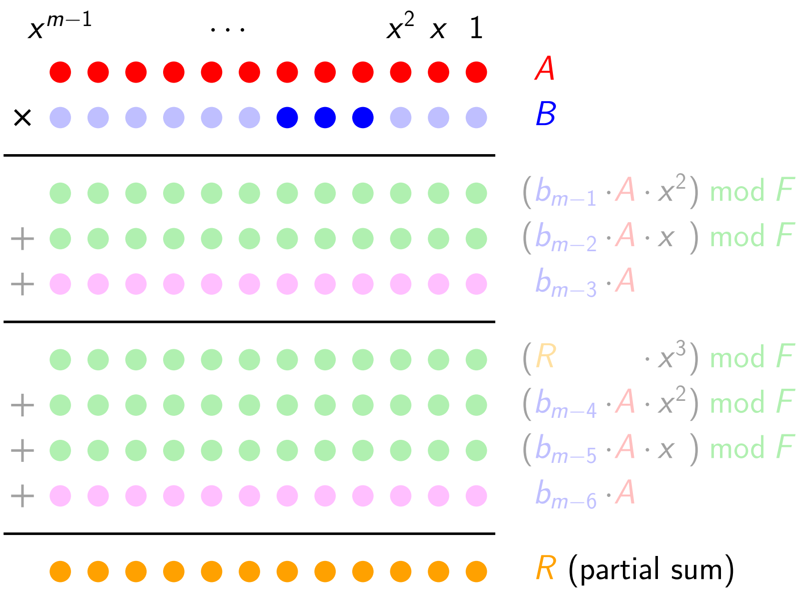

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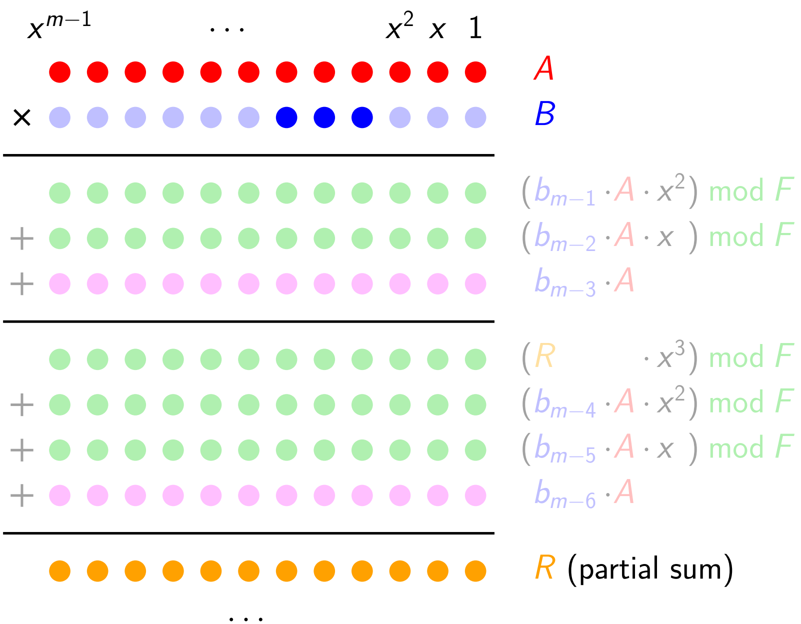
  - $\lceil m/d \rceil$ clock cycles for computing the product
  - area grows with d: area–time trade-off



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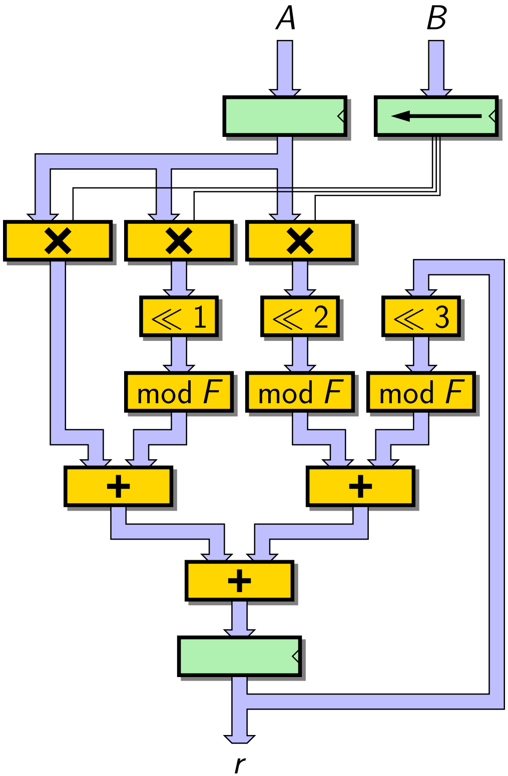

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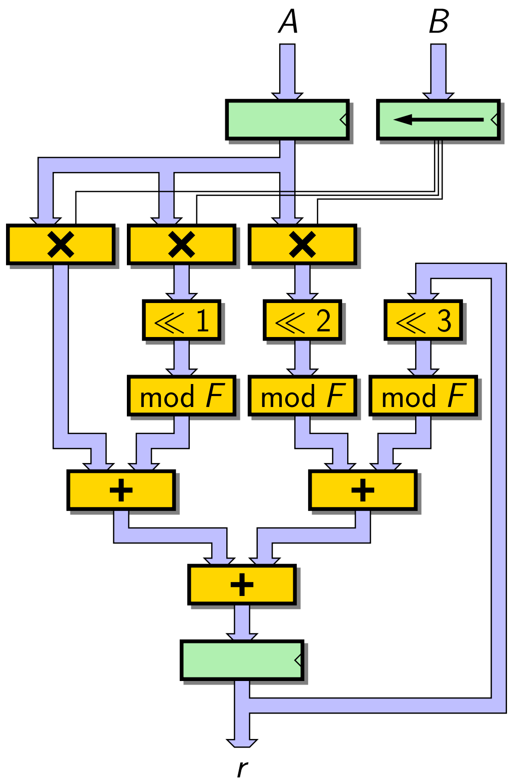
- feedback loop for accumulation of the result



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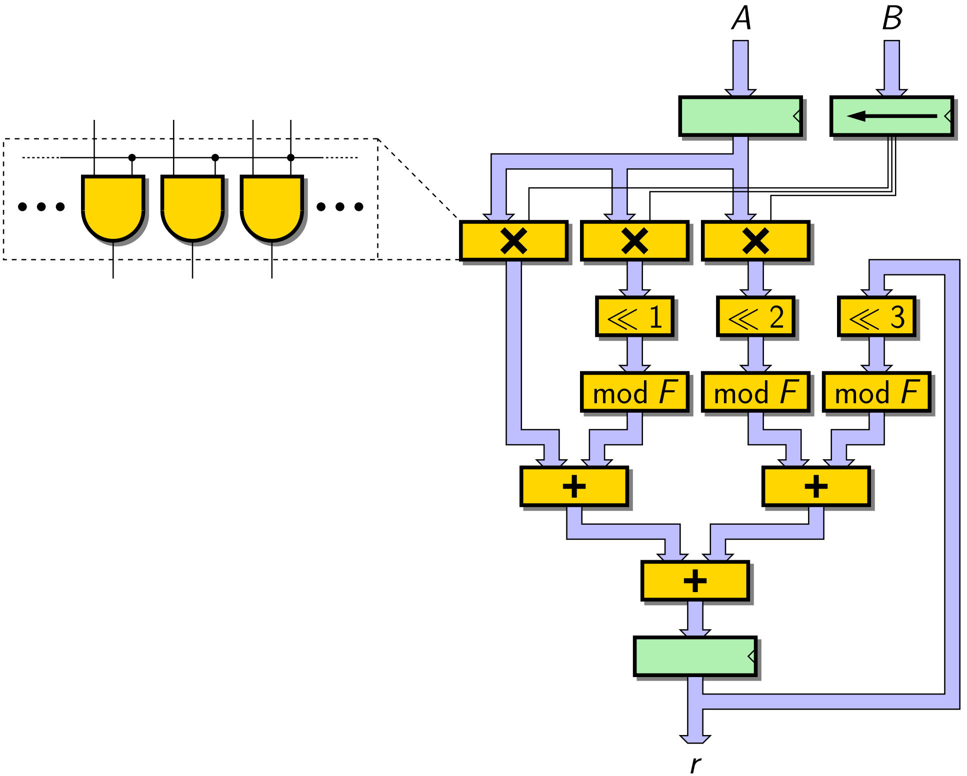

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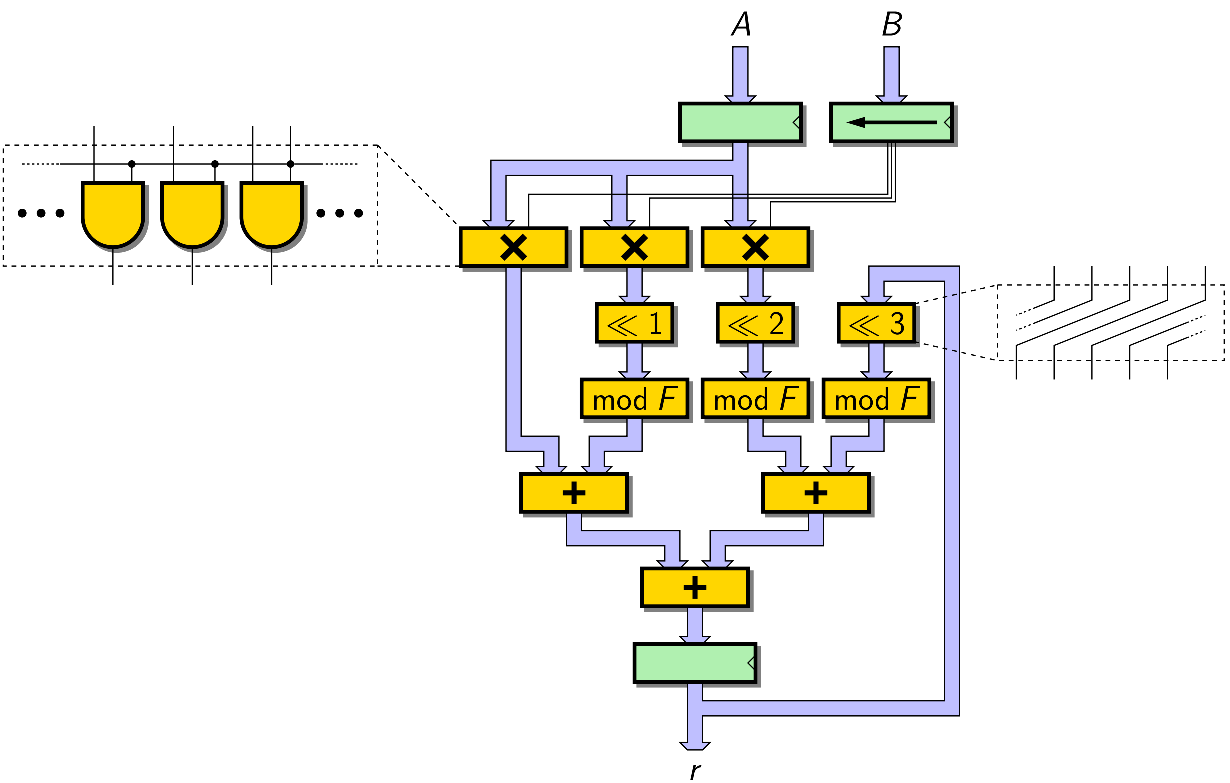

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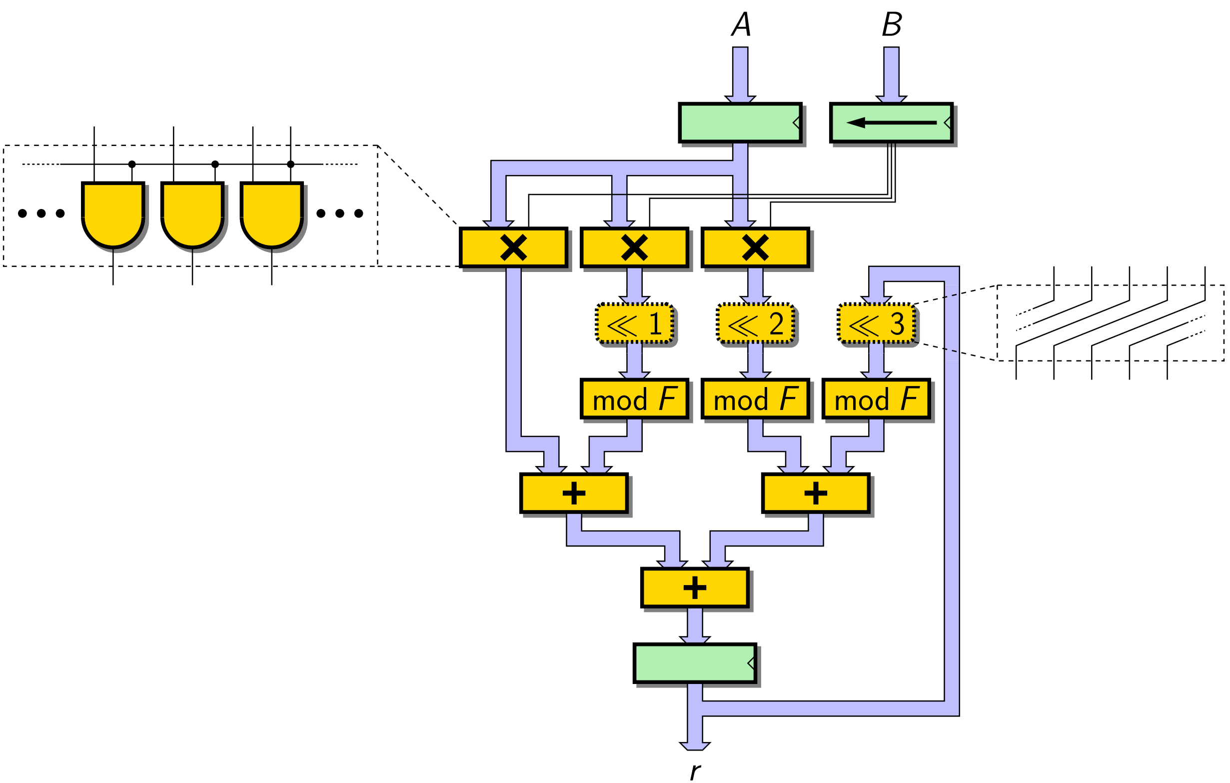

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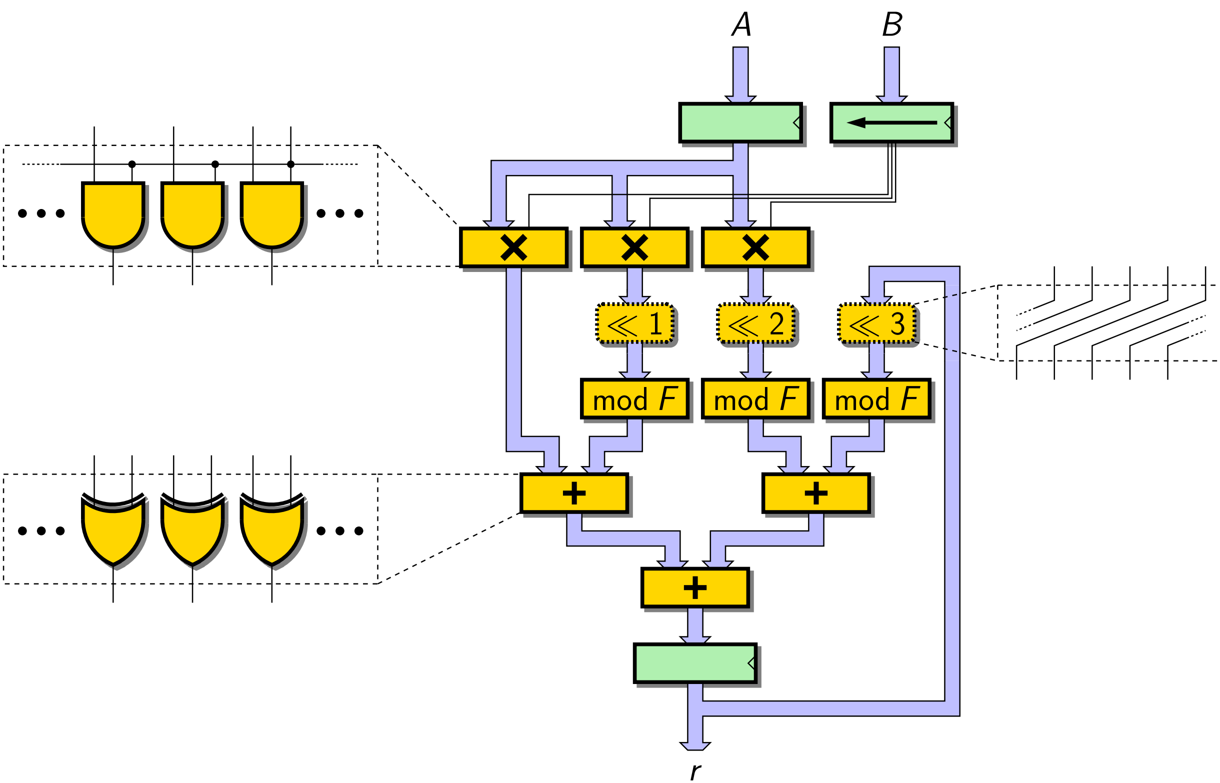

Arithmetic coprocessor for ECC over $\mathbb{F}_{2^m}$

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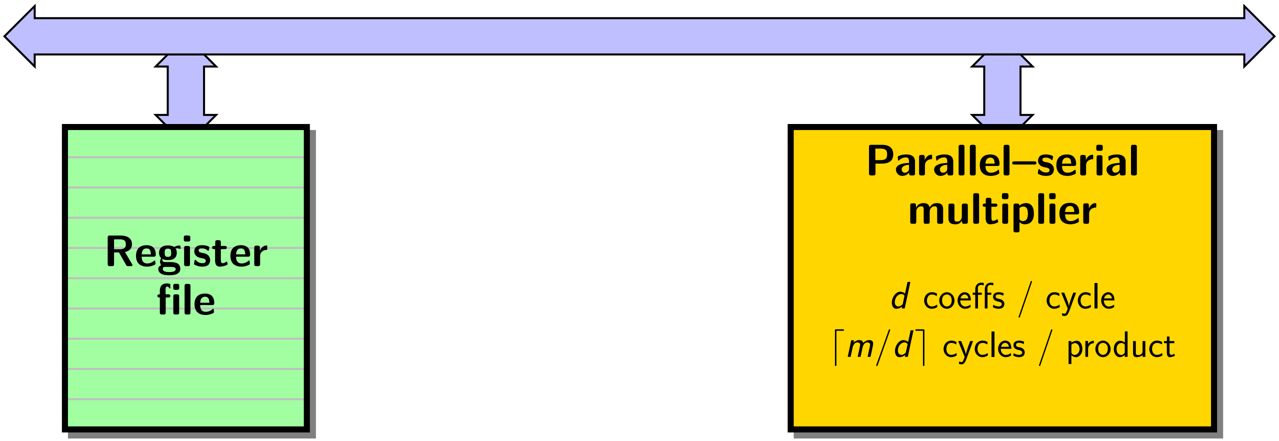

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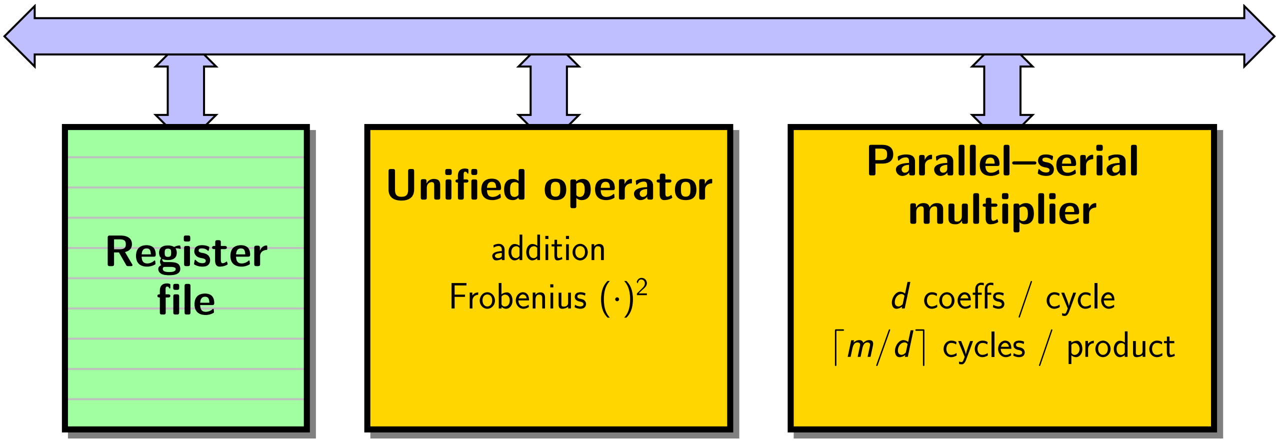

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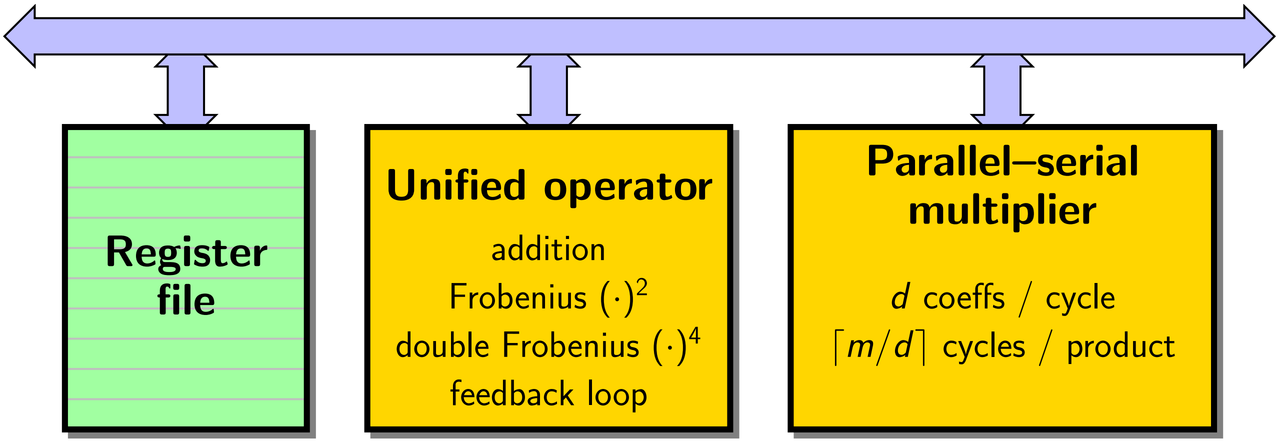
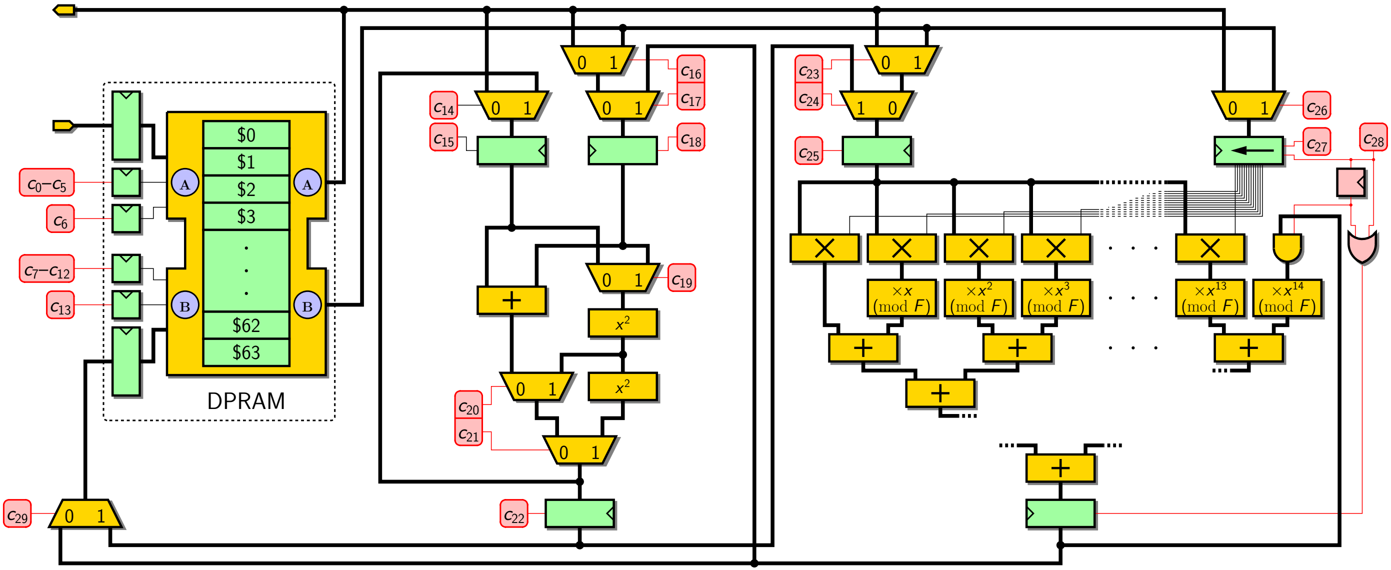


Arithmetic coprocessor for ECC over $\mathbb{F}_{2^m}$



Thank you for your attention

Questions?